THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and thirty-eighth regular meeting of the Society was held in New York City on Saturday, April 25, 1908, extending through the usual morning and afternoon sessions. The attendance included the following thirty-five members:

Professor G. A. Bliss, Professor Maxime Bôcher, Professor F. N. Cole, Professor L. P. Eisenhart, Professor G. H. Hallett, Mr. G. W. Hartwell, Dr. A. A. Himowich, Professor E. V. Huntington, Professor J. I. Hutchinson, Professor Edward Kasner, Professor C. J. Keyser, Mr. E. H. Koch, Jr., Mr. W. C. Krathwohl, Professor P. A. Lambert, Dr. G. H. Ling, Mr. L. L. Locke, Mr. E. B. Lytle, Dr. Emory McClintock, Professor Max Mason, Mr. A. R. Maxson, Dr. R. L. Moore, Professor Richard Morris, Professor W. F. Osgood, Mr. H. W. Reddick, Mr. E. W. Sheldon, Mr. L. P. Siceloff, Professor S. E. Slocum, Mr. F. H. Smith, Professor P. F. Smith, Dr. C. E. Stromquist, Professor H. D. Thompson, Mr. C. A. Toussaint, Professor Oswald Veblen, Professor H. S. White, Professor J. W. Young.

The President of the Society, Professor H. S. White, occupied the chair, being relieved at the afternoon session by Professor C. J. Keyser. The Council announced the election of the following persons to membership in the Society: Professor H. E. Buchanan, Lincoln College, Lincoln, Ill.; Mr. E. F. A. Carey, University of California; Professor F. E. Chapman, Southern University, Greensboro, Ala.; Professor R. C. Maclaurin, Columbia University; Mr. E. J. Miles, University of Chicago; Mr. C. A. Stiles, University Preparatory School, Ithaca, N. Y.; Mr. J. S. Thompson, Mutual Life Insurance Company, New York, N. Y.; Mr. O. A. Turney, Phoenix, Ariz.; Mr. C. B. Walsh, Ethical Culture School, New York, N. Y.; Professor R. T. Wilbur, Christian Brothers College, St. Louis, Mo.; Miss E. R. Worthington, Yale University. Ten applications for admission to membership in the Society were received.

At the meeting of the Council Professor E. B. Van Vleck was reelected a member of the Editorial Committee of the *Transactions*, to serve for the three years 1908–1911. It was

decided to hold the summer meeting and colloquium of the Society in 1909 at Princeton University, and Professors Fine, Osgood, Holgate, and the Secretary were appointed a committee to make the appropriate arrangements. A committee consisting of Professor P. F. Smith, C. J. Keyser, and G. A. Bliss was appointed to consider the question of holding the next annual meeting of the Society at Baltimore in affiliation with the American association for the advancement of science.

The following papers were read at this meeting:

(1) Professor S. E. SLOCUM: "The collapse of tubes under external pressure."

(2) Professor E. B. WILSON: "On the differential equations of the equilibrium of an inextensible string."

(3) Professor E. B. WILSON: "On the principle of relativity."

(4) DR. ELIJAH SWIFT: "Note on the second variation in an isoperimetric problem."

(5) Professor J. I. HUTCHINSON: "The hypergeometric functions of n variables."

(6) Professor EDWARD KASNER: "Note on Meusnier's theorem."

(7) Professor B. F. FINKEL: "Determination of the groups of order 2^m which contain selfconjugate cyclic subgroups of order 2^{m-4} and whose generating operations correspond to the partitions $[m-4, 4]$, $[m-4, 3, 1]$."

(8) Professor J. W. YOUNG: "Two-dimensional chains and the classification of complex collineations in a plane."

(9) Mr. C. N. MOORE: "On certain constants analogous to Fourier's constants."

(10) Professor PAUL SAUREL: "On the distance from a point to a surface."

(11) Mr. E. B. LYTLE: "Multiple integrals over iterable fields."

(12) Professor P. A. LAMBERT: "The fundamental theorem of algebra."

(13) Professor E. V. HUNTINGTON and Mr. H. P. FORTÉ: "On the fluctuations in the speed of a flywheel."

(14) Professor E. V. HUNTINGTON: "On the theory of the gyroscope, with special reference to the Brennan monorail car" (preliminary communication).

(15) Dr. O. E. GLENN: "Studies in the theory of degenerate algebraic curves" (preliminary communication).

In the absence of the authors the papers of Professor Wilson, Dr. Swift, Professor Finkel, Mr. Moore, Professor Saurel, and Dr. Glenn were read by title. The papers of Dr. Swift and Mr. Moore appeared in the May BULLETIN. Professor Saurel's paper will be published in the July number of the BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Professor Slocum's paper is an analysis of experimental data on the collapse of tubes under external pressure. The material for this analysis is found in the recent extensive experiments made independently by Professor Carman and Professor Stewart.

The rational formulas for thin and thick tubes are discussed, and the significance of the elastic constants involved, as indicating the nature of collapse, is pointed out. The fact that the physical properties of the material, given in connection with the experiments in question, did not include these elastic constants, indicates the irrational nature of the results and their consequent limitations. The characteristic difference between the failure of thin and thick tubes is also explained.

The nature and amount of the deviation of commercial tubes from the ideal assumptions upon which the rational formulas are based is examined and corrections determined. In this connection the flexibility of rational formulas as applying to particular cases, and the rigidity of empirical formulas as merely striking an average, are contrasted in terms of numerical percentages.

A graphical comparison of the curves represented by the rational formulas for thin and thick tubes is made for steel and for brass. The limit for use is determined as the abscissa of the point of intersection of these curves, and is found to agree closely with the empirical values determined experimentally.

A graphical comparison is also made between Stewart's empirical formulas and the corrected rational formulas for thin and thick tubes, the former representing an ellipse and its linear tangent, and the latter a cubic curve and a parabola respectively.

The results of the analysis are embodied in the formulas

$$P = C\frac{2E}{1-m^2}\left(\frac{t}{D}\right)^3 \text{ for thin tubes,}$$

$$P = 2u_c K \frac{t}{D}\left(1 - K\frac{t}{D}\right) \text{ for thick tubes,}$$

where E = Young's modulus, m = Poisson's ratio, u_c = ultimate compressive strength, P = external collapsing pressure, t = thickness, D = diameter, and C and K = correction constants, all dimensions being expressed in inches and pounds as units. C and K are so expressed that they may be determined independently for individual specimens, or for the various types of commercial tubes. From the data under discussion C and K are found to have the following values: For lap welded steel boiler flues, $C = .69$ and $K = .89$; for cold drawn seamless steel flues, $C = .76$; for drawn seamless brass tubes, $C = .78$.

The article will appear in full in the *Engineering News*.

2. Professor Wilson treats the differential equations of the equilibrium of an inextensible string from the point of view of Lie's theory of groups. He obtains the conditions that the equations should admit a group of eight parameters, and thereby determines a case in which the equations are completely integrable by quadratures and where the field of force is neither central nor parallel. The paper will be offered to the *Transactions*.

3. In a short note, Professor Wilson discusses the principle of relativity in electrodynamics with reference to recent suggestions and recent experimental developments. The paper will be offered to the *Philosophical Magazine*.

5. The $n + 1$ linearly independent hypergeometric integrals regarded as functions of n parameters x_i entering into the integrand are considered in the paper by Professor Hutchinson. The chief problem solved is the determination of the group of homogeneous linear transformations which these integrals undergo when the x_i describe closed paths. It is also shown that a certain hermitian form is invariant for this group.

6. Professor Kasner shows that the most general system of space curves for which Meusnier's theorem is valid is defined by a differential equation of the type $\alpha y'' + \beta z'' + \gamma = 0$, where the coefficients are arbitrary functions of x, y, z, y', z'. This includes Lie's extension as a special case. Such systems arise in connection with the general transformation of lineal elements.

8. Fundamental in the classification of the linear fractional transformations on one complex variable — *i. e.*, of the projectivities on a complex line — is the notion of a chain of points on the line. A chain may be defined as any class of points projective with the real points of the line. If the complex numbers be represented in the usual manner by the real points of a plane, the chains of the complex line are represented by the circles of this plane. The behavior of the projectivities on the line with reference to the chains of the line then leads readily to the well-known classification of these projectivities into hyperbolic, elliptic, etc.*

The corresponding classification of the projective transformations in a complex space of n dimensions must lead first to a generalization of the notion of a chain. It is obtained naturally as follows: Using homogeneous coordinates, any point in a complex space S_n of n dimensions is represented by $(x_0, x_1, x_2, \cdots, x_n)$, where the x_i are any complex numbers (the combination $(0, 0, \cdots, 0)$ of course excluded). The totality of such points in which the x_i are any real numbers forms the real subspace R_n of n dimensions in S_n. Any subspace of S_n obtained by subjecting R_n to any linear homogeneous transformation on the x_i with complex coefficients Professor Young calls an n-dimensional chain in S_n, or more briefly an n-chain.† In the present paper Professor Young considers in detail the case $n = 2$. Certain fundamental properties of 2-chains in a complex plane are first developed, which are then applied to effect the desired classification "with respect to reality" of the complex collineations in a plane. Each of the well-known five types of collineations in the plane is subdivided with reference to the existence of invariant 2-chains, and the existence of each subtype is established. This classification is of interest not merely in the field of projective geometry and continuous groups, but also in the theory of automorphic functions of two variables. The treatment is largely synthetic and the proofs are so simple as to make the methods applicable to larger values of n with comparatively little increase in complexity.

11. Former discussions of the problem of the reduction of

* For a synthetic treatment of the one-dimensional case, cf. a paper by the author presented to the Society at its last (February) meeting on "The geometry of chains on a complex line."

† For a non-analytic definition of this notion, cf. "A system of assumptions for projective geometry," by O. Veblen and J. W. Young, presented to the Society at its last Christmas meeting.

multiple integrals to iterated integrals have been restricted to functions defined over Jordan measurable fields. Mr. Lytle defines iterable fields in such a way as to include all measurable and a large class of non-measurable aggregates. By means of this new class of iterable aggregates important relations between multiple and iterated integrals are derived. These relations lead to a very general condition for inversion of iterated integrals.

12. The purpose of Professor Lambert's paper is to present a general method for determining all the roots of any algebraic equation by means of infinite series. The method consists in forming three algebraic functions of x from the given equation (1) $f(y) = 0$, (a) by introducing a factor x into all the terms of (1) except the first and last; (b) by introducing a factor x into all the terms of (1) except the first and second; (c) by introducing a factor x into all the terms of (1) except the second and last.

These algebraic functions are expanded into power series in x by Laplace's series. If in these power series x is made unity, the resulting series, if convergent, determine the roots of the given equation. It is shown that all the roots of the algebraic equation can be expressed in infinite series derived either from the algebraic function formed in accordance with (a), or from the two algebraic functions formed in accordance with (b) and (c).

13. The paper of Professor Huntington and Mr. Forté contains an exact formula for the small fluctuations in the speed of the fly wheel of a horizontal engine, with the results of illustrative experiments made in the Harvard engineering laboratory by the use of a tachograph.

Let N be the number of revolutions per minute, regarded as a function of the crank angle θ, and suppose that $N = N_1$ when $\theta = \theta_1$. Then the value of N for any other value of θ is given by

$$N^2 = N_1^2 \left\{ 1 + \frac{\dfrac{1800g}{\pi^2 N_1^2 r}\left(\dfrac{U}{W_p r}\right)\Bigg|_{\theta=\theta_1}^{\theta=\theta} - \left(A^2 + \dfrac{w}{W_p}B^2\right)\Bigg|_{\theta=\theta_1}^{\theta=\theta}}{\dfrac{W}{W_p}\dfrac{K^2}{r^2} + A^2 + \dfrac{w}{W_p}B^2} \right\},$$

where A and B are numerical functions of θ depending on the

proportions of the engine, and U is the total work done by the external forces (steam pressure, load, gravity, and friction) during the interval from $\theta = 0$ to $\theta = \theta$. Here W, W', w are the weights of flywheel, piston, and connecting rod, K = radius of gyration of the flywheel, and r = crank radius. The values of A and B are as follows:

$$A = \frac{\sin(\theta + \phi)}{\cos\phi}, \text{ where } \phi \text{ is given by } \sin\phi = \frac{r}{l}\sin\theta;$$

$$B^2 = \sin^2\theta\left[1 + \frac{l-c}{l}\frac{r\cos\theta}{l\cos\phi}\right]^2 + \frac{c^2}{l^2}\cos^2\theta + \frac{k^2}{r^2}\left(\frac{r}{l}\frac{\cos\theta}{\cos\phi}\right)^2.$$

Here l = length of connecting rod, c = distance from wrist pin to center of gravity, and k = radius of gyration about the center of gravity. For the value of U, if the effective pressure P on the piston and the resisting moment RL of the load L about the shaft are known functions of θ, and if the expression for the work of friction is assumed to be of the form $(F + fL)R$, where F and f are constants, then

$$U = \int_0^\theta [PdX - RLd\theta - (F + fL)Rd\theta] - wc\sin\phi,$$

where $X = r(1 - \cos\theta) + l(1 - \cos\phi)$, so that $dX = rAd\theta$.

The formula for N holds good for all conditions of pressure and load; the condition for "uniform speed" is that the integral in the expression for U shall reduce to zero at the end of each revolution.

14. Professor Huntington's second paper, when completed, will contain approximate solutions of the equations of motion which present themselves in the theory of the gyrostatic balancing of the Brennan monorail car.

15. The general subject of the relations which exist between an algebraic curve and those curves of lower degree into which it can decompose has been studied only incidentally, or at best only in connection with special curves of low degrees. Certain results have been published by Cayley, Zeuthen, Taylor, and others; but their work is only fragmentary. Dr. Glenn has organized a general theory of the decomposition of curves of nth degree, and presents, in the form of a preliminary report approximately half of the results of his investigation.

The report contains chapters on the following topics: the asymptotic properties of ternary homogenous forms; the penultimate forms of a curve; the existence of "free summits" on a degenerate algebraic curve; the classification of curves on the basis of the penultimate forms. Further investigations are in progress. The memoir, as a whole, will be offered for publication the coming year.

F. N. COLE,
Secretary.

THE APRIL MEETING OF THE CHICAGO SECTION.

THE twenty-third regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago, on Friday and Saturday, April 17–18, 1908. One session was held on Friday and two on Saturday. On Friday evening twenty of the members dined together informally in the private dining room of the University of Chicago Commons. The attendance at the three sessions included forty-five persons, among whom were the following thirty-three members:

Dr. G. D. Birkhoff, Professor Oscar Bolza, Dr. R. L. Börger, Professor H. E. Buchanan, Mr. Thomas Buck, Professor W. H. Bussey, Dr. A. R. Crathorne, Professor D. R. Curtiss, Professor L. E. Dickson, Mr. Arnold Dresden, Mr. E. P. R. Duval, Professor W. B. Ford, Mr. R. M. Ginnings, Professor Harriet E. Glazier, Professor C. N. Haskins, Mr. T. H. Hildebrandt, Mr. F. H. Hodge, Dr. A. C. Lunn, Mr. W. D. MacMillan, Professor G. A. Miller, Dr. J. C. Morehead, Professor F. R. Moulton, Dr. L. I. Neikirk, Professor H. L. Rietz, Miss Ida M. Schottenfels, Mr. A. R. Schweitzer, Professor H. E. Slaught, Dr. Clara E. Smith, Dr. A. L. Underhill, Professor E. B. Van Vleck, Dr. A. E. Young, Professor J. W. A. Young, Professor Alexander Ziwet.

Professor G. A. Miller, Vice-President of the Society and chairman of the Section, presided at all of the sessions. In opening the meeting he spoke of the great loss sustained by the Society in the recent death of Professor Heinrich Maschke. A committee, consisting of Professors E. B. Van Vleck, Alexander Ziwet, and H. E. Slaught, was appointed to draft suitable resolutions on behalf of the Section, and these were pre-

sented at the close of the session on Saturday morning, adopted by a rising vote, and ordered spread upon the minutes. The resolutions are printed immediately following this report. A memorial of Professor Maschke, prepared by Professor Bolza, will soon appear in the BULLETIN.

The following papers were read:

(1) Dr. C. H. SISAM: "On a locus determined by concurrent tangents."

(2) Professor W. B. FORD: "On the integration of the equation $a_0(x)\,u(x+2)+a_1(x)\,u(x+1)+a_2(x)\,u(x)=0$."

(3) Professor D. R. CURTISS: "On the real branches of implicit functions in the neighborhood of multiple points."

(4) Mr. L. L. DINES: "A method of investigating numbers of the forms $6^\kappa \cdot s \pm 1$."

(5) Professor L. E. DICKSON: "Criteria for the irreducibility of a reciprocal equation."

(6) Professor L. E. DICKSON: "On reciprocal abelian equations."

(7) Professor L. E. DICKSON: "On the congruence $x^n+y^n+z^n \equiv 0 \pmod{p}$."

(8) Professor JACOB WESTLUND: "Note on the equation $x^n+y^n=nz^n$."

(9) Mr. F. H. HODGE and Mr. E. J. MOULTON: "On certain characteristics of orbits for a general central force."

(10) Professor G. A. MILLER: "The central of a group."

(11) Dr. A. E. YOUNG: "On the problem of the spherical representation and the characteristic equations of certain classes of surfaces."

(12) Dr. A. C. LUNN: "A continuous group related to von Seidel's optical theory."

(13) Dr. A. C. LUNN: "A minimal property of simple harmonic motion."

(14) Dr. A. C. LUNN: "The deduction of the electrostatic equations by the calculus of variations."

(15) Mr. A. R. SCHWEITZER: "Remark on Enriques's review of the foundations of geometry."

(16) Mr. A. R. SCHWEITZER: "On the calculi of relations, classes, and operations."

(17) Professor E. J. WILCZYNSKI: "Projective differential geometry of curved surfaces, fourth memoir."

(18) Dr. G. D. BIRKHOFF: "Irregular integrals of ordinary linear differential equations."

(19) Professor R. D. CARMICHAEL: "On the general tangent to plane curves."

(20) Professor R. D. CARMICHAEL: "On plane algebraic curves symmetrical with respect to each of two rectangular axes."

(21) Professor O. D. KELLOGG: "Note on the geometry of continuously turning curves."

(22) Dr. I. SCHUR: "Beiträge zur Theorie der Gruppen linearer homogener Substitutionen."

(23) Mr. W. D. MACMILLAN: "On the character of the solutions of homogeneous linear equations with periodic coefficients."

(24) Mr. A. R. SCHWEITZER: "On the quaternion as an operator in Grassmann's extensive algebra."

Mr. Dines was introduced by Professor Curtiss. Dr. Schur's paper was communicated to the Society through Professor Dickson. In the absence of the authors the papers of Dr. Sisam, Professor Westlund, Professor Wilczynski, Professor Carmichael, and Dr. Schur were read by title. Professor Dickson's first paper appears in the present number of the BULLETIN. His second paper has been published in the April number of the *American Mathematical Monthly*. Abstracts of the other papers follow below, the numbering being the same as that attached to the titles in the above list.

1. In the plane of an arbitrary algebraic curve C there exists a second curve C' such that the points of tangency of three of the tangents to C from each point of C' are collinear. Dr. Sisam's paper is devoted to the discussion of the locus C' and of the dual locus.

2. Professor Ford's paper considers the behavior for large values of x of the general solution of the equation

$$a_0(x)\,u(x+2) + a_1(x)\,u(x+1) + a_2(x)\,u(x) = 0,$$

the coefficients $a_0(x)$, $a_1(x)$, $a_2(x)$ being given functions (real or complex) of x defined for all positive integral values of x sufficiently large. Especial attention is given to the important case in which the coefficients are developable in Maclaurin series about the point $x = \infty$. The results obtained may be employed in the study of the convergence of algebraic continued fractions, as will appear subsequently. The paper has been offered for publication in the *Transactions*.

3. In the paper of Professor Curtiss conditions for the existence of real branches of an implicit function $f(x, y) = 0$, in the neighborhood of a multiple point, are obtained by methods which are analogous to those used by Stolz and others in the theory of maxima and minima, and which constitute a generalization of the method of Dini for an ordinary point. The invariant character of these conditions is discussed, and the connection of semi-definite cases with the contact of branches of the curves

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$$

is indicated.

4. Mr. Dines's paper describes a method of determining factors of numbers of the forms $6^\kappa \cdot s \pm 1$ or primes of those forms, easily applicable to numbers above the limits of the existing factor tables. The x_κ and x'_κ functions of any prime p are defined respectively as the least positive solutions of the congruences $6^\kappa \cdot s - 1 \equiv 0 \pmod{p}$ and $6^\kappa \cdot s + 1 \equiv 0 \pmod{p}$, and recursion formulas are developed for computing the functions. By means of a table of these functions for successive primes, it is possible to deplete by a sieve process analogous to the "sieve of Eratosthenes," any series of consecutive integers, so that any integer remaining, if substituted for s will make $6^\kappa \cdot s + 1$ (or $6^\kappa \cdot s - 1$) prime. The paper will be offered to the *Annals of Mathematics*.

7. In the congruence $x^n + y^n + z^n \equiv 0 \pmod{p}$, considered by Professor Dickson, n and p are assumed to be odd prime numbers. As there are obviously solutions prime to p when $p - 1$ is not divisible by n and when $p - 1$ is a multiple of $3n$, it is assumed that $p = mn + 1$, where m is not divisible by 3. In the second supplement to his Théorie des Nombres, Paris, 1825, Legendre has treated the cases $m = 2, 4, 8, 10, 14, 16$, each by a separate discussion; but overlooked the exceptional character of $n = 3$ when $m = 10$ or 14. Professor Dickson develops a method which applies, without a separation of cases, to the values of m considered by Legendre, and obtains the following result. The congruence has no set of solutions prime to p when n and $p = mn + 1$ are odd primes and $m = 2, 4, 8, 10, 14, 16, 20, 22, 26, 28, 32, 40, 56, 64$, apart from the following exceptions: $m = 10, 14, 20$, $n = 3$; $m = 22$, $n = 3, 31$; $m = 26$, $n = 3, 5$; $m = 28$, $n = 7$; $m = 32$, $n = 3$; $m = 40$,

$n=7$; $m=56$, $n=5$, 11, 17, 113, 227; $m=64$, $n=3$, 7, 229, 337, 757. The results are applied to Fermat's last theorem to show that $x^n+y^n+z^n=0$ has no integral solutions, each not divisible by n, for n an odd prime <1700, the previous limits being 100 (Sophie Germain), 200 (Legendre), 223 (Maillet), 257 (Mirimanoff). The method is due to Sophie Germain and requires the determination of a prime $p=mn+1$ for which the above congruence has no solutions prime to p and such that p is not a divisor of m^m-1. The latter number was completely factored by Legendre for the values m which he treated. Part of this work may be avoided. For example, if m is a multiple of 4, the quadratic reciprocity theorem shows that a prime divisor $mn+1$ (n odd) of m^m-1 must divide $m^{\frac{1}{2}m}-1$. Additional reduction of the exponent may be made when m is a power of 2.

In the second part of the paper, n is given a fixed value and it is proposed to determine the primes $p=mn+1$ for which the congruence has no integral solutions prime to p. A general method of attack is developed and the analysis completed for the cases $n=3$, 5, 7, the number of values of p being 2, 4, 4 respectively.

The first part has been offered for publication in the *Messenger of Mathematics* and the second part in *Crelle's Journal*.

8. In Professor Westlund's note it is proved that the equation $x^n+y^n=nz^n$ is not solvable in rational integers if n is any odd prime and z is not divisible by n.

9. The orbits of particles moving under central forces varying as the first power and as the inverse second, third, and fifth powers have been discussed in detail by many writers, and other special powers also have received attention. The paper of Mr. Hodge and Mr. Moulton considers certain general characteristics of orbits for forces varying as any integral power of the distance, by discussing a transformation of the energy integral expressed in polar coordinates. Without knowing the equations of the orbits, the existence of apses is shown, the aphelian and perihelian distances are found, and certain other properties are deduced.

It is found that for attractive forces the law of the inverse second power gives an intermediate case between two general classes of orbits; the one where the orbits pass through either

the center of force or infinity; the other where the orbits have finite perihelion and aphelion distances greater than zero. The paper will be published in the *American Mathematical Monthly.*

10. The totality of the invariant operators of any group G constitutes a characteristic subgroup K known as the central, or the cogredient subgroup of G. When G is abelian, it coincides with K, and when G contains only one invariant operator, K is identity. In these extreme cases the concept of a central does not simplify the considerations with respect to G, but in all other cases it is useful. The quotient group of G with respect to K is the group of cogredient isomorphisms of G and is represented by I_1. The main object of Professor Miller's paper is to determine some of the properties of G from known properties of I_1. The following theorems are proved: If I_1 involves an invariant operator of order n, G contains an invariant commutator of order n and it also has a cyclic quotient group of this order. Moreover, the order of G is divisible by n^3, and I_1 involves two independent operators of order n. The number of Sylow subgroups of any order in G is exactly the same as the number of the Sylow subgroups of the corresponding order in I_1. When all the Sylow subgroups of I_1 are cyclic, every abelian subgroup of I_1 corresponds to an abelian subgroup of G and vice versa. If the order of such an I_1 is $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m}$, where p_1, $p_2, \cdots, p_m$ are distinct primes in descending order of magnitude, and if I_1 contains a cyclic subgroup of order $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_\gamma^{\alpha_\gamma}$, then G contains a characteristic cyclic subgroup of the same order. The paper will be offered for publication in the *Transactions.*

11. For any surface S referred to lines of curvature the equations of Gauss and Codazzi may be written in the form

$$\frac{\partial M}{\partial v} - \frac{\partial N}{\partial u} = PQ, \quad \frac{\partial P}{\partial v} + MQ = 0, \quad \frac{\partial Q}{\partial u} - NP = 0,$$

where the expression for the corresponding linear element on the gaussian sphere is

$$d\sigma^2 = P^2du^2 + Q^2dv^2.$$

Any other equation involving one or more of the functions, taken with the three equations above, determines a certain system of lines on the sphere. We may say therefore that this new equation is characteristic of these lines, or rather of the

system of surfaces which have them as the spherical representation of their lines of curvature.

From this standpoint Dr. Young has considered the systems of surfaces defined by the following equations:

$$(1)\ M=0,\quad (2)\ M+N=0,\quad (3)\ \frac{\partial M}{\partial u}+\frac{\partial N}{\partial v}=0,$$

$$(4)\ \frac{\partial^2 \log M}{\partial u\,\partial v}+MN=0,\quad (5)\ \frac{\partial^2 \log MK}{\partial u\,\partial v}-K=0,$$

where

$$K=-\frac{\partial^2 \log M}{\partial u\,\partial v}-MN,\ \text{etc.}$$

12. In this paper Dr. Lunn shows how von Seidel's paraxial equations for a centered optical instrument may be interpreted as defining a two-parameter transformation which generates a three-parameter group of Lie's first type. It is shown that all invariants of this group are functions of the one which occurs in a known formula due to Lagrange and Helmholtz.

13. In his second paper Dr. Lunn points out that the rectilinear oscillation of a particle is determined as simple harmonic by the condition that the dissipation of energy under a resistance proportional to the velocity shall be a minimum when the period and mean square displacement are assigned. The conjugate points bounding the interval of true minimum are determined.

14. Starting with Maxwell's definitions of volume and surface density of charge, and of density of energy, in terms of electric displacement and strength of field, Dr. Lunn gives in this paper a deduction of the conditions of electrostatic equilibrium without assuming the existence of the potential function, which on the contrary appears as a lagrangian indeterminate multiplier.

15. Mr. Schweitzer's note refers to a remark of Enriques in the Encyclopädie der Mathematischen Wissenschaften, volume III, 1, page 34. For the definition of euclidean geometry on the basis of the descriptive-projective geometry, we distinguish, in the well-known manner, between the proper and improper points of the space and agree that all improper points shall lie on the unique improper plane. Every elliptic polar system Σ_0 in this improper plane gives rise to a euclidean geometry $G(\Sigma_0)$.

Thus by means of the class of elliptic polar systems $\{\Sigma_0\}$ in the improper plane is defined a unique class of euclidean geometries $\{G(\Sigma_0)\}$. The members of this latter class are, however, given by the geometry function $G(\Sigma)$ of a variable elliptic polar system Σ in the improper plane. This geometry function is then uniquely defined and consists of a complete set of euclidean propositions in which the congruence propositions appear as implications, having as terms propositional functions of the variable Σ. (Compare Russell, The Principles of Mathematics, pages 13, 82.)

16. As formal types of relational, class, and operational propositions Mr. Schweitzer takes, aRb, $a\epsilon C(b)$, $O_b(a)$. These propositions may be considered as expressing, respectively, "a possesses the relation R with reference to b," "a is in the class C with reference to b," "the operation O with reference to b affects a." It is assumed that aRb implies $a\epsilon C(b)$ implies $O_b(a)$. The following analysis is studied:

$$aRb = aR + b = a + Rb,$$
$$a\epsilon C(b) = aC + b = a + Cb,$$
$$O_b(a) = bO + a = b + Oa,$$

where $aR \neq Ra$, etc., and the symbol of composition "$+$" is an indefinable which implies no ordering of the terms.

17. In his previous papers on the general projective theory of surfaces, Professor Wilczynski assumed that the surface was referred to its asymptotic curves. The object of the present memoir is to find the general expressions for the invariants and covariants if the parameters to which the surface is referred are arbitrary. In this way it becomes possible to apply the results of the author, as developed in his earlier papers, to a given surface without first determining its asymptotic curves. The paper will be offered for publication in the *Transactions*.

18. Dr. Birkhoff effects a classification of irregular integrals by proving that all such integrals are linearly expressible in terms of certain canonical irregular integrals; the coefficients in the linear form are analytic at the singular point $x = a$ in question. These canonical irregular integrals are of the form $x^s\phi(x)$, where $\phi(x)$ is analytic except at $x = a$, and are fully

characterized by the specification of a certain finite set of constants. By means of a method wherein certain integral equations are used, it is found possible to make a direct study of the general irregular integrals whose characteristic properties depend on the aforesaid constants. With the aid of these results it is finally proved that with every irregular integral are associated certain functional equations. Such a functional equation for e^x is $e^{x+c} = e^c \cdot e^x$.

19. The differential calculus furnishes an easy method of finding the general tangent to any curve whose equation is in cartesian coordinates. In this paper Professor Carmichael obtains the same formula for algebraic curves without the aid of the calculus, and extends this result so as to apply to a large number of cases in which the curve is transcendental. The reasoning is of such elementary nature as to be readily understood by a student in his first course. And it is believed that this general formula may be introduced to the pupil's notice with profit. The paper will be offered for publication in the *American Mathematical Monthly*.

20. In his second paper Professor Carmichael points out the form of the cartesian equation of plane algebraic curves symmetrical with respect to each of two rectangular axes, and classifies such curves of the fourth degree, the grouping in classes being determined by certain geometric properties common to those of each class. The paper will be offered for publication in the *American Mathematical Monthly*.

21. In a recent article in the *Transactions of the American Mathematical Society*, Professor Kellogg found it necessary to borrow two facts from the geometry of continuously turning curves; his note presents the proofs of these facts. It will be published in the *American Mathematical Monthly*.

22. In his paper in the *Transactions*, volume 6 (1905), page 504, Professor Loewy transforms any linear group G into a group the matrix of whose general substitution is a compound matrix with square matrices $A_{11}, A_{22}, \cdots, A_{\mu\mu}$ in the main diagonal, and with zero matrices to the right of the diagonal; further, the totality of matrices A_{11} form a (maximal) completely reducible group, the A_{22} a completely reducible group, etc. This normalization may be accomplished by a transfor-

mation within a given domain Ω containing the coefficients of the substitutions of G. The groups $A_{11}, A_{22}, \cdots$ completely reducible in Ω, are uniquely determined by G and Ω, when equivalent (conjugate) groups are considered to be identical. The paper by Dr. Schur shows that, when equivalent groups are considered to be not distinct, the groups $A_{11}, A_{22}, \cdots$ are entirely independent of the choice of the domain Ω, so that, if $A'_{11}, A'_{22}, \cdots, A'_{\mu'\mu'}$ are the analogous groups for a new domain Ω' likewise containing the coefficients of the substitutions of G, then $\mu' = \mu$ and A'_{ii} is equivalent to A_{ii}. Of the various additional theorems established by Dr. Schur, mention may be made of the following: Every linear group G irreducible in a domain Ω is completely reducible in the domain Z of all numbers; if the most general matrix commutative with G has exactly r distinct characteristic roots, G decomposes in Z into r irreducible groups $F, F_1, \cdots$ of equal degree; these r groups may be represented in r conjugate algebraic domains $\Omega(\rho_i)$. If exactly l of these r groups are non-equivalent, then r/l is an integer which divides the degree of the group F. The paper will be offered for publication in the *Transactions*.

23. The character of the solutions of homogeneous linear differential equations with periodic coefficients (period 2π) is well known from the writings of Floquet, Callandreau, and others. The purpose of Mr. MacMillan's paper is to exhibit similarly the character of the solutions when the equations are linear with periodic coefficients but non-homogeneous. It is found that, when the characteristic exponents are all distinct and none of them congruent to zero (mod $\sqrt{-1}$) and the non-homogeneous terms have the period 2π, the particular solutions are periodic with the period 2π. If one of the characteristic exponents is congruent to zero (mod $\sqrt{-1}$), then in general the particular solution will contain terms involving the independent variable as a linear factor, that is to say the solution is not periodic.

If the non-homogeneous terms of the differential equations contain terms of the form $\sin(i+\lambda)t$ or $\cos(i+\lambda)t$ (where i is an integer, and λ is any real number) then the particular solution depending upon these terms is trigonometric, provided the characteristic exponents are all distinct and not congruent to zero nor equal to $\lambda\sqrt{-1}$. When these conditions are not

fulfilled the solution in general is not periodic. The appropriate discussion for these cases is given.

24. In the *Mathematische Annalen* volume 12, page 376, Grassmann has defined the middle product of two vectors as a "formal" sum of their outer and inner products, namely, $AB=[AB]+\lambda[A/B]$, the product being associative if $\lambda=-1$ and non-associative if $\lambda=+1$. Mr. Schweitzer observes that the quaternion and its extension occur as operators in the grassmannian vector space of 2^n dimensions ($n=1, 2, 3, \cdots$). For example, for $n=2$, if

$$X=x_1E_1+x_2E_2+x_3E_3+x_4E_4,$$

the vectors $i'x$, $j'x$, $k'x$ are suitably defined and the vectors $i''/\lambda'\cdot X$, $j''/\lambda''\cdot X$, $k''/\lambda'''\cdot X$, where λ', λ'', λ''' are constants different from zero, are determined by the specification that the outer product $X\cdot i'X\cdot j'X\cdot k'X$ shall be identically equal to $\rho(x_1^2+x_2^2+x_3^2+x_4^2)^2E_1E_2E_3E_4$, where ρ is a constant. These vectors may also be determined by a simple tactical construction of a square matrix of 2^n elements. It is found then that as operators on the vector X,

$$\left(\frac{i''}{\lambda'}\right)^2=\left(\frac{j''}{\lambda''}\right)^2=\left(\frac{k''}{\lambda'''}\right)^2=-1,\quad \frac{i''}{\lambda'}\cdot\frac{j''}{\lambda''}=-\epsilon\frac{k''}{\lambda'''},$$

$$\frac{i''}{\lambda'}\cdot\frac{j''}{\lambda''}\cdot\frac{k''}{\lambda'''}=\epsilon,$$

where $\epsilon^2=1$. The quaternion Q is defined as an operator $Q(X', X)$ which transforms a grassmannian vector X into the vector X'. For the fundamental unit vectors E_1, E_2, E_3, E_4 the following relations are valid: (1) If $E_{r_1}E_{r_2}$ is the grassmannian complement of $E_{r_3}E_{r_4}$, then $Q(E_{r_1}E_{r_2})=\epsilon Q(E_{r_3}E_{r_4})$. (2) If E_{r_1}, E_{r_2}, E_{r_3} are any three unit vectors, then

$$Q(E_{r_1}E_{r_2})\cdot Q(E_{r_2}E_{r_3})=Q(E_{r_1}E_{r_3}).$$

(3) If $E_{r_1}\neq E_{r_2}$, then $Q(E_{r_1}E_{r_2})=-Q(E_{r_2}E_{r_1})$. The generalization of these relations is discussed.

Thus, from the operative viewpoint, the quaternion appears in a real and homogeneous manner; also what may be called the operative sum and product of two grassmannian vectors in 2^n space is readily defined.

H. E. SLAUGHT,
Secretary of the Section.

HEINRICH MASCHKE.

Resolutions on the Death of Professor Heinrich Maschke.

Whereas, in the death of Professor Heinrich Maschke the Chicago Section of the American Mathematical Society suffers the loss of one of its most honored, influential, and beloved members, the undersigned committee appointed by the Section hereby expresses its deep appreciation of the services and character of Professor Maschke.

From the first organization of the Chicago Section until the present session Professor Maschke has been one of its most active and inspiring members. By his genial qualities, his unusual sympathy as a teacher, his integrity and intellectual honesty, he has won and held the affection of those who have known him. By his ability as an investigator he has contributed greatly to the development of productive mathematical scholarship in the formative period of the Society, and in his own person he has exemplified the influence of German scholarship which has contributed so potently to this development. In the death of Professor Maschke the Section for the first time feels the loss of one of its leaders.

The committee recommends that this resolution be incorporated in the records of the Section and that a copy be transmitted to Mrs. Maschke as an expression of our sympathy in her deep loss.

E. B. Van Vleck,
Alexander Ziwet,
H. E. Slaught.

CRITERIA FOR THE IRREDUCIBILITY OF A RECIPROCAL EQUATION.

BY PROFESSOR L. E. DICKSON.

(Read before the Chicago Section of the American Mathematical Society, April 17, 1908.)

1. A reciprocal equation $f(x) = x^m + \cdots = 0$ is one for which

$$x^m f(1/x) \equiv cf(x).$$

Replacing x by $1/x$, we see that $f \equiv c^2 f$, $c = \pm 1$. Now $f(x)$ has the factor $x \pm 1$ and hence is reducible, unless m is even and $c = +1$. Further discussion may therefore be limited to equations

$$(1)\quad F(x) \equiv x^{2n} + c_1 x^{2n-1} + c_2 x^{2n-2} + \cdots + c_2 x^2 + c_1 x + 1 = 0$$

of even degree and having

$$(2)\qquad x^{2n} F(1/x) \equiv F(x).$$

Let R be a domain of rationality containing the c's.

Under the substitution

$$(3)\qquad x + 1/x = y,$$

$x^{-n} F(x)$ becomes a polynomial in y,

$$(4)\qquad \phi(y) = y^n + k_1 y^{n-1} + \cdots + k_n,$$

with coefficients in R. By a suitable choice of the c's, the k's may be made equal to any assigned values.

We shall establish in §§ 2–7 the following:

THEOREM. *Necessary and sufficient conditions for the irreducibility of $F(x)$ in the domain R are*

(I) *$\phi(y)$ must be irreducible in R.*

(II) *$F(x)$ must not equal a product of two distinct irreducible functions of degree n.*

The second condition is discussed in §§ 8–10.

2. The irreducibility of $F(x)$ in R implies that of $\phi(y)$. For, if

$\phi(y) = (y^l + \cdots)(y^m + \cdots), \quad (l + m = n, \quad l > 0, \quad m > 0),$

then would

$$F(x) = x^l\{(x + 1/x)^l + \cdots\} \cdot x^m\{(x + 1/x)^m + \cdots\}.$$

3. If $F(x)$ has in R an irreducible factor

$$A(x) = x^{2r+1} + a_1x^{2r} + \cdots + a_{2r}x + a_{2r+1}$$

of odd degree, then $F(x)$ has the irreducible factor.

$$B(x) \equiv \frac{x^{2r+1}}{a_{2r+1}} A\left(\frac{1}{x}\right) = x^{2r+1} + \frac{a_{2r}}{a_{2r+1}} x^{2r} + \cdots + \frac{a_1}{a_{2r+1}} x + \frac{1}{a_{2r+1}},$$

not identical with $A(x)$. For, from $F \equiv AQ$ and (2) follows

$$F(x) \equiv B(x)Q', \quad Q' \equiv a_{2r+1}x^{2n-2r-1}Q(1/x).$$

Next, if $B \equiv A$, then

$$A = x^{2r+1} \pm 1 + a_1x(x^{2r-1} \pm 1) + a_2x^2(x^{2r-3} \pm 1) + \cdots + a_rx^r(x \pm 1),$$

so that A would have the factor $x \pm 1$ and be reducible.

4. If $\phi(y)$ is irreducible in R, $F(x)$ has in R no irreducible factor A of odd degree $< n$. For, if so, $P \equiv AB$, where B is given in § 3, would be a self-reciprocal factor of $F(x)$. In fact,

$$P(1/x) = A(1/x)A(x)/x^{2r+1}a_{2r+1}, \quad x^{2(2r+1)}P(1/x) = AB = P(x).$$

Hence, in view of (3), $x^{-(2r+1)}P(x)$ would equal a factor of degree $2r + 1$ of $\phi(y)$.

5. If $F(x)$ has in R an irreducible factor

$$A(x) = x^{2r} + a_1x^{2r-1} + \cdots + a_{2r-1}x + a_{2r}$$

of even degree, then $F(x)$ has the irreducible factor

$$B(x) = \frac{1}{a_{2r}} x^{2r} A\left(\frac{1}{x}\right).$$

If $B(x) \equiv A(x)$, A is self-reciprocal, viz.,

$$A(x) = x^{2r} + 1 + a_1(x^{2r-1} + x) + \cdots + a_{r-1}(x^{r+1} + x^{r-1}) + a_rx^r.$$

In fact, the conditions for $B \equiv A$ are

$$a_{2r} = \pm 1, \quad a_{2r-1} = \pm a_1, \quad a_{2r-2} = \pm a_2, \quad \cdots.$$

For the lower signs, A has the factor $x^2 - 1$, contrary to hypothesis.

6. If $\phi(y)$ is irreducible in R, $F(x)$ has in R no irreducible factor A of even degree $< n$. For, by § 5, either B is distinct from A so that AB is a self-reciprocal factor of $F(x)$, or else A itself is a self-reciprocal factor. In either case $\phi(y)$ would have in R a factor of degree $< n$.

7. It follows from §§ 4, 6 that, when $\phi(y)$ is irreducible in R, $F(x)$ has no irreducible factor of degree $< n$. Further, by §§ 3, 5, an irreducible factor $A(x)$ of degree n implies a second irreducible factor $x^n A(1/x)$, algebraically distinct from $A(x)$. The theorem of § 1 is therefore proved.

8. It remains to consider the case $F = AB$,

$$A = x^n + a_1 x^{n-1} + \cdots + a_n,$$

$$B = x^n + \frac{a_{n-1}}{a_n} x^{n-1} + \cdots + \frac{a_1}{a_n} x + \frac{1}{a_n},$$

where A and B are distinct irreducible functions in R. To determine the a_i, we have n distinct relations

$$(5) \quad a_1 + a_{n-1}/a_n = c_1, \quad a_2 + (a_1 a_{n-1} + a_{n-2})/a_n = c_2, \cdots$$

We may eliminate $a_1, \cdots, a_{n-1}$ and obtain an equation for a_n. As shown in § 9, this equation is of degree 2^n. Except for certain sets of values of the c_i, we may express $a_1, \cdots, a_{n-1}$ rationally in terms of a_n; the problem is then reduced to the consideration of the rationality of a root of the equation of degree 2^n. This equation for a_n is a reciprocal equation. In fact, if we set

$$A_1 = a_{n-1}/a_n, \quad \cdots, \quad A_{n-1} = a_1/a_n, \quad A_n = 1/a_n,$$

equations (5) become

$$(5') \quad A_1 + A_{n-1}/A_n = c_1, \; A_2 + (A_1 A_{n-1} + A_{n-2})/A_n = c_2, \cdots.$$

That the equations (5′) are throughout of the same form as equations (5) is evident from the fact that we have merely interchanged the rôles of the factors A and B of F. Hence the equation in a_n, obtained by eliminating $a_1, \cdots, a_{n-1}$ from

(5), is identical with the equation in $A_n = 1/a_n$, obtained from (5′).

9. Denote the roots of $F = 0$ by

$$\alpha_1,\quad \alpha_1^{-1},\quad \alpha_2,\quad \alpha_2^{-1},\quad \cdots,\quad \alpha_n,\quad \alpha_n^{-1}. \tag{6}$$

A factorization $F = AB$, of the kind considered in § 8, corresponds uniquely to a separation of the roots (6) into two sets each of n roots, such that reciprocal roots belong to different sets. Hence the roots of the first set may be selected in

$$\frac{2n(2n-2)(2n-4)\cdots 2}{n!} = 2^n$$

ways. The number of factors A is thus 2^n.

10. For $n = 2$, we set $\alpha_1 = \alpha$, $\alpha_2 = \beta$, and have

$$\alpha+\beta+\alpha^{-1}+\beta^{-1}=c_1,\quad 2+\alpha\beta^{-1}+\beta\alpha^{-1}+\alpha\beta+\alpha^{-1}\beta^{-1}=c_2. \tag{7}$$

From these we derive

$$\alpha^2+\beta^2+\alpha^{-2}+\beta^{-2}=c_1^2-2c_2.$$

Hence $\alpha\beta+\alpha^{-1}\beta^{-1}$ and $\alpha\beta^{-1}+\alpha^{-1}\beta$ are the roots of

$$z^2-(c_2-2)z+c_1^2-2c_2=0. \tag{8}$$

The quartic for a_2(§ 8) is obtained by setting

$$z = a_2 + a_2^{-1}. \tag{9}$$

By (7_1), $\alpha+\beta$ is a rational function of $\alpha\beta$ and c_1 when $c_1 \neq 0$. Hence, for $c_1 \neq 0$, the necessary and sufficient conditions for the factorization $F = AB$ in R are that the roots

$$z_\pm = \tfrac{1}{2}(c_2-2) \pm [(1+\tfrac{1}{2}c_2)^2 - c_1^2]^{\frac{1}{2}} \tag{10}$$

of (8) be rational and that one of the values $(z_\pm^2-4)^{\frac{1}{2}}$ be rational, so that (9) shall lead to a rational value of a_2. Incorporating the condition that (4) shall be irreducible in R, we obtain the

THEOREM.* *The necessary and sufficient conditions that*

$$x^4+c_1x^3+c_2x^2+c_1x+1 \quad (c_1 \neq 0)$$

* For other proofs by the writer, see *Amer. Math. Monthly*, vol. 10 (1903), p. 221; vol. 15 (1908), p. 75. The first paper cited also treats reciprocal sextic equations.

shall be irreducible in a domain R are that $(c_1^2 - 4c_2 + 8)^{\frac{1}{2}}$ *be irrational, and that either* $l = [(1 + \frac{1}{2}c_2)^2 - c_1^2]^{\frac{1}{2}}$ *be irrational or else l rational and* $[\frac{1}{2}c_2^2 - c_1^2 - 2 \pm (c_2 - 2)l]^{\frac{1}{2}}$ *both irrational.*

11. The only linear fractional transformations which replace a reciprocal equation by a reciprocal equation are

$$x' = \pm \frac{\alpha x + \beta}{\beta x + \alpha} \qquad (\alpha^2 \neq \beta^2). \tag{11}$$

Then y, given by (3), undergoes the transformation

$$y' = \pm \frac{(\alpha^2 + \beta^2)y + 4\alpha\beta}{\alpha\beta y + \alpha^2 + \beta^2}. \tag{12}$$

The transformation on $\frac{1}{2}y$ is the square of (11).

UNIVERSITY OF CHICAGO,
March, 1908.

A NEW GRAPHICAL METHOD FOR QUATERNIONS.

BY PROFESSOR JAMES BYRNIE SHAW.

(Read before the Southwestern Section of the American Mathematical Society, November 30, 1907.)

1. ANY quaternion q may be written in the form $q = (w + xi) + (y + zi)j$. For convenience let us represent numbers of the form $w + xi$ (practically equivalent to ordinary complex numbers save in their products by j) by Greek characters, so that q may be written

$$q = \alpha + \beta j,$$

where for any number β we have $\beta j = j\bar{\beta}$, $\bar{\beta}$ being the conjugate of β.

The tensor of q is then the square root of the sum of the squares of the moduli of α, β. Also the scalar of q is $\frac{1}{2}(\alpha + \bar{\alpha})$, that is, the real part of α.

2. The product of $q = \alpha + \beta j$ and $r = \gamma + \delta j$ is

$$qr = (\alpha\gamma - \beta\bar{\delta}) + (\alpha\delta + \beta\bar{\gamma})j,$$

and also we have

$$rq = (\alpha\gamma - \bar{\beta}\bar{\delta}) + (\bar{\alpha}\delta + \beta\gamma)j.$$

The sum of q and r is $q + r = \alpha + \gamma + (\beta + \delta)j$.

3. The graphical representation is as follows: Assume two independent planes, the A-plane and the B-plane, on which we represent α and β like ordinary complex numbers. These representations are the A and B projections of q, respectively. The sum of q and r has for its projections the sums of the respective projections of q and r. To represent the projections of the product qr, we find by the usual methods the products $\alpha\gamma$, $\beta\bar{\delta}$, $\alpha\delta$, $\beta\bar{\gamma}$. To find $\alpha\gamma$, γ is turned through the angle of α, then extended in the ratio $T\alpha : 1$. So for the others. We then add vectorially $\alpha\gamma$ and $-\beta\bar{\delta}$; $\alpha\delta$ and $\beta\bar{\gamma}$.

4. If we consider r as fixed, and let the multiplier q be such that the extremities of its projections describe straight lines, then the extremities of the projections of qr will describe straight lines. For if

$$\alpha = \lambda_1 + t(\lambda_2 - \lambda_1), \quad \beta = \mu_1 + t(\mu_2 - \mu_1),$$

then

$$qr = [\lambda_1\gamma + t(\lambda_2 - \lambda_1)\gamma - \mu_1\bar{\delta} - t(\mu_2 - \mu_1)\bar{\delta}] + [\lambda_1\delta + t(\lambda_2 - \lambda_1)\delta + \mu_1\bar{\gamma} + t(\mu_2 - \mu_1)\bar{\gamma}]j.$$

The extremity of the A projection of qr thus describes the line

$$\rho = \lambda_1\gamma - \mu_1\bar{\delta} + t[(\lambda_2 - \lambda_1)\gamma - (\mu_2 - \mu_1)\bar{\delta}],$$

which passes through the points $\lambda_1\gamma - \mu_1\bar{\delta}$ and $\lambda_2\gamma - \mu_2\bar{\delta}$. The extremity of the B projection of qr describes the line

$$\rho = \lambda_1\delta + \mu_1\bar{\gamma} + t[(\lambda_2 - \lambda_1)\delta + (\mu_2 - \mu_1)\bar{\gamma}],$$

which passes through the points $\lambda_1\delta + \mu_1\bar{\gamma}$ and $\lambda_2\delta + \mu_2\gamma$. Further if $\alpha = e^{int}\alpha_1$, $\beta = e^{int}\beta_1$, then

$$qr = e^{int}(\alpha_1\gamma - \beta_1\bar{\delta}) + e^{int}(\alpha_1\delta + \beta_1\gamma)j.$$

Thus the projections of the product will describe circles with the same period as those of q.

5. We have $S.qr = 0$ when $S(\alpha\bar{\gamma} + \beta\delta) = 0$, that is

$$\alpha\bar{\gamma} + \beta\delta = ti.$$

Solving for β we have for any value of α,

$$\beta = -\frac{\alpha\bar{\gamma}}{\delta} + t\frac{i}{\delta}.$$

That is, for any value of α the values of β terminate on a straight line perpendicular to $\bar{\delta}$. Likewise for any value of β the values of α terminate on the line

$$\alpha = -\frac{\beta\delta}{\gamma} + t\frac{i}{\gamma},$$

which is perpendicular to γ. This is equivalent to saying that if β terminates on the line through ξ at right angles to $\bar{\delta}$, α will terminate on the line through $-\xi\delta/\gamma$ perpendicular to γ. Or in brief, to the line through θ/δ perpendicular to $\bar{\delta}$ corresponds the line through $-\theta/\gamma$ perpendicular to γ, in the sense that any quaternion q whose projections terminate on these two lines in the A and B planes respectively, is *perpendicular* to the quaternion $r = \gamma + \delta j$.

6. The applications of this method to four-dimensional space are obvious, the A and B planes being two planes having only one point in common, the origin. The representation is in fact an adaptation of the descriptive geometry of four-dimensional space to the representation of quaternions as four-dimensional vectors, although this interpretation is not essential to the representation.

LOGIC AND THE CONTINUUM.

BY PROFESSOR EDWIN BIDWELL WILSON.

THE problem whether every set and in particular the continuum can be well ordered has attracted considerable attention since the days when G. Cantor first stated it in 1883.* In 1904 Zermelo offered an affirmative solution of the problem,† but his solution has not been generally regarded with favor and the discussion of the whole problem has now turned largely to a discussion of his solution. In a recent article he has summarized and discussed this discussion so fully that no repetition is called for at this time.‡ In entering so vast an arena of conflict, I would make no pretense of settling the dif-

* *Mathematische Annalen*, vol. 21, p. 550.

† Zermelo, "Beweis, dass jede Menge wohlgeordnet werden kann," *Mathematische Annalen*, vol. 59, pp. 514–516.

‡ Zermelo, "Neuer Beweis für die Möglichkeit einer Wohlordnung," *ibid.*, vol. 65 (1907), pp. 107–128.

ficulties or of solving the problem; I merely desire to call attention to some general matters of logic which have an application to this problem and to others, and which appear to me to have been somewhat neglected. The introduction of a different point of view may be of service. My comments will at first be directed upon the idea of the categoricity or sufficiency of a set of postulates.

Although the method of nominal definition from logical elements is regarded in many quarters as preferable to the determination by means of postulates, when a specified field of mathematics is to be investigated, there still remains so much difficulty in the way of the purely nominal definition of mathematics that the determination by means of postulates cannot as yet be considered as obsolete, perhaps not even as obsolescent; and this method apparently must obtain in laying the foundations of logic itself, whether or not the nominal definition is finally adopted for the subsidiary sciences. The ideas that a set of postulates should not be redundant and must not be contradictory have been widely exploited. There is, however, another idea which so far as I am aware was first clearly stated and emphasized by Huntington and which has not yet received a great measure of recognition. This idea is that a set of postulates may be sufficient or categorical.*

A set of postulates $[P] = P_1, P_2, \cdots, P_n$ connecting a set of undefined symbols $[S] = S_1, S_2, \cdots, S_m$ is said to be categorical if between the elements of any two assemblages, each of which independently contains the undefined symbols $[S]$ and satisfies the postulates $[P]$, it is possible to set up a one to one correspondence which preserves the significance of the undefined symbols. This last restriction is vital — it ensures a sort of logical conformality in the correspondence. For it is ob-

* Huntington, "A complete set of postulates for the theory of absolute continuous magnitude, *Transactions of the American Mathematical Society*, vol. 3 (1902), pp. 264–279. "Complete sets of postulates for the theories of positive integral and positive rational numbers," *ibid.*, pp. 280–284. "Complete sets of postulates for the theory of real quantity," *ibid.*, vol. 4 (1903), pp. 358–370. In these earlier articles Huntington uses the term sufficient; in later communications, "A set of postulates for real algebra, comprising postulates for a one-dimensional continuum and for the theory of groups," *ibid.*, vol. 6 (1905), pp. 17–41, and "The fundamental laws of addition and multiplication in elementary algebra," *Annals of Mathematics*, ser. 2, vol. 8 (1906), pp. 1–44, he adopts somewhat hesitatingly the term categorical from a paper of Veblen's and points out, *Annals*, p. 26, that Peano had and apparently realized that he had a categorical set of postulates for positive integers as early as 1891.

vious that if any set of elements $[K]$ which satisfies the postulates $[P]$ on the symbols $[S]$ may be put into a one to one relation with a set of elements $[K']$, it must be possible to interpret $[P]$ and $[S]$ in the set $[K']$ by a mere transference from the original set $[K]$. This procedure is of frequent occurrence in mathematics,* but it has little to do with categoricity for the reason that $[K]$ and $[K']$ are not determined independently.

It is not always desirable and indeed not always possible to obtain a set of postulates which shall be categorical: for it may well happen that the systems to be determined are such that not even a one to one correspondence between their elements is available, to say nothing of the preservation of the interpretation of the symbols $[S]$. For instance in defining a group it would clearly be unwise so to restrict the definition that all groups were of equal multitude. And in defining a geometry in a space of constant curvature, there is little to be gained by a restriction which specifies the magnitude of the curvature unless it be zero. It is, however, advisable to have a categorical determination of euclidean geometry and of the continuum and of real algebra. This has been accomplished by Huntington and Veblen.† In case these domains of mathematics are defined nominally it is equally important that the definition shall be sufficiently detailed to ensure an equivalent precision in the system defined.

Huntington gives a subsidiary definition or explanation of the idea of categoricity wherein he asserts that if a set $[P]$ of postulates on the undefined symbols $[S]$ is categorical, then every proposition concerning $[S]$ must be deducible from the postulates $[P]$ or be in contradiction with them.‡ In interpreting this statement it should be noted that not every conceivable property of the defined set $[K]$ is asserted as either deducible from $[P]$ or in contradiction with $[P]$, but only such properties as are expressible solely in terms of $[S]$. Thus the statement that the rational numbers greater than 0 are red and those less than 0 are blue is neither in contradiction with

* A little reflection will show that even here the value of the transference of properties from $[K]$ to $[K']$ is almost nil except as the transferred properties may be related to properties which $[K']$ independently possesses.

† References to Huntington have been given. Veblen, "A system of axioms for geometry." *Transactions of the American Mathematical Society*, vol. 5 (1904), pp. 343–384.

‡ See his last paper cited, or Veblen's.

Huntington's set of postulates nor deducible from them, because the statement contains terms (red, blue) which are not defined or nominally definable in the system; moreover, any statement about "alternate rational numbers," where alternate refers to the numbers in their natural order, is neither compatible nor incompatible with the postulates, because that statement is not a proposition at all but merely a meaningless collocation of words.*

I have advisedly used the terms compatible with $[P]$ and incompatible with $[P]$ in the respective places of deducible from $[P]$ and in contradiction with $[P]$. Is this a justifiable change? It is readily seen that any proposition (statement which may be either true or false but not meaningless) phrased in terms of $[S]$ or derived symbols must be either compatible with $[P]$ or incompatible with $[P]$: for, owing to the categoricity of the determination, it is impossible that the proposition should be true of one set $[K]$ and false of another set $[K']$ both of which satisfy the postulates. Does it follow that every such proposition must either be deducible from $[P]$ or in contradiction with $[P]$? This question, this suggestion that compatibility and deducibility may not be the same when applied to categorically determined systems is vital in logic and requires careful discussion.†

In the first place it is evident that if the determination of a system is not categorical, then compatibility and deducibility cannot be entirely equivalent. This follows from the very idea of compatibility, to wit, that any proposition phrased in terms of the undefined symbols shall either be true for all the systems determined by $[P]$ or be false for them all. Any individual postulate of a set $[P]$ which is not redundant is neither compatible nor incompatible (or is both compatible and incompatible, if pre-

* The difficulties that are involved in meaningless statements have been recognized since very early times; for instance, the case of the liar who says he is a liar. Such is also in all probability the difficulty with the class of all classes; see Zermelo, "Untersuchungen über die Grundlagen der Mengenlehre," *Mathematische Annalen*, vol. 65 (1908), pp 261–281.

† Huntington quotes a remark of Davis on this subject in a later paper "A set of postulates for ordinary complex algebra." *Transactions*, vol. 6 (1905), p. 210, footnote †; Veblen dismisses it with the statement that any additional postulate would have to be considered redundant even were it not deducible from the categorical set by a finite number of syllogisms, *loc. cit.*, p. 346. As I do not know what he means by this, I am unable to say whether or not his position is in practical accord with that which I shall set forth in the following paragraphs.

ferred) with the other postulates of $[P]$.* In the second place consider a system determined categorically. Here every proposition phrased in terms of $[S]$ is either compatible or incompatible with $[P]$. What, however, does the word deducible mean? The meaning is entirely relative to the system of logic which is available for drawing conclusions from the set of primitive propositions $[P]$. Some may consider that the human mind has instinctively at its disposal all valid methods of deduction. This is a tremendous postulate, and one entirely devoid of other than sentimental value. In fact, if it leads to the abandoning of the research for valid methods of deduction, it is dangerous and worse than useless. It is an essential of the modern attitude in logic that the deducer should state distinctly his form of inference. Hence deducible cannot be regarded as equivalent to compatible.

It is clear that in an ideal perfection of logic compatibility and deducibility would be equivalent for categorically defined systems. That state of perfection appears at present to be very remote. The constant attempt to bring compatibility and deducibility into coincidence will undoubtedly do much to advance the condition of logic, just as in the days before Huntington so clearly stated the idea of categoricity that attempt did much to advance mathematics.† It appears to me, however, that it may be a distinct gain in precision and hence a considerable advantage to admit the following orienting propositions, namely:

So long as there is an unsolved problem of pure mathematics the solution may be lacking

1° because the class of objects to which the problem belongs is not sufficiently determined or

2° because the available logical methods of deduction are insufficient; but in case the class of objects is categorically determined, 2° alone applies.‡

To take some examples. Fermat's problem of the solution of

$$x^n + y^n = z^n, \quad n > 2,$$

* This statement is capable of a small amount of generalization by including under proper restrictions theorems as well as postulates.

† For instance, the development of noneuclidean geometry could be interpreted as a somewhat unconscious striving in this direction.

‡ There is, of course, the possibility that the lack of solution is due to our failure to perceive the method by which the logical principles already in our possession should be combined to reach the desired result. Logically, however, this is a very different sort of failure from either of those mentioned, and it has been thought better not to refer to it in the text.

in terms of integers cannot fail of solution because of insufficient determination of the class of integers, inasmuch as integral algebra has been established upon a categorical basis. From the nature of this problem it appears likely that the lack of solution is not due to the incompleteness of the methods of deduction, but rather to a failure properly to combine the existing methods. Goldbach's theorem that any even number may be written as the sum of two primes cannot, for a similar reason, fail of proof owing to an insufficiency of determination. Peano states,* if I understand his inflexionless latin, that this theorem fails of a satisfactory demonstration because, like the theorem that any infinite set contains an enumerable subset, it depends on the application of the principle of Zermelo (see below). Thus for Peano and Zermelo deducible has different meanings; and any one is at liberty to side with either party to the dispute.† It may, however, be pointed out that, by some method of combining forms of deduction which both would admit, Goldbach's theorem might perhaps be demonstrated: but it is difficult to see how such a reconciliation could be operated in regard to the theorem on the existence of an enumerable subset in any infinite set.

In discussing the problem of the well-ordering of the continuum from the point of view I have here set forth, it is first necessary to ask if the definition of well-ordering can be phrased in terms of the undefined symbols introduced in the categorical determination of the continuum. On comparing Zermelo's definition of well-ordering and Huntington's set of postulates,‡ it appears that the definition can be so phrased without introducing new symbols other than the general logical constants introduced in framing the postulates. Hence Zermelo has no right to add any new postulate expressing a property of the continuum. As a matter of fact, the postulate that he does add concerns classes in general and the usage he makes of it and his own statements show that what he has done is to demand a new postulate or principle of logic. Thus to prove

* Peano, "Supra theorema de Cantor-Bernstein," *Revista de Matematica*, vol. 8 (1906), pp. 136–157; see especially p. 146.

† As Zermelo, in common with most other investigators, does not state what his logical postulates are, it is impossible to say just what deducible does mean for him. This difficulty, which is scarcely felt at all in ordinary reasoning, becomes very troublesome when one tries to follow the great number of different writers on transfinites.

‡ Zermelo, *loc. cit.*, *Annalen*, 65, p. 111. Huntington, *loc. cit.*, *Transactions*, 6, p. 31.

that the proposition that the continuum is well ordered is compatible with the postulates concerning the continuum, he enlarges his logic and shows that any set is well ordered.

In view of the facts that it may be doubted whether our logic is yet complete and that Zermelo's postulate is apparently not in contradiction with the other logical postulates, it is difficult to see how any one can deny him the right to proceed as he does. The only question appears to be whether his method is valuable. And certainly, if such an elementary theorem of arithmetic as Goldbach's theorem cannot be proved or disproved without some such addition as Zermelo makes to our logical procedure, it seems as if some such addition were imperative; for however one may hesitate about committing himself to the intricacies of the continuum, he surely would not indefinitely consign Goldbach's theorem to the class of non-deducibles. It therefore is proper to examine the new principle of logic and its consequences.

The first statement of Zermelo's principle is: For an infinite ensemble of sets it is possible to find a correspondence which correlates to each set a single element of that set.* For a finite ensemble of sets the proposition is readily admitted. The same is true of the later formulation of the postulate: A set S which is divided into subsets $A, B, C, \cdots$ each containing at least one element but containing no elements in common contains at least one subset S_1 which has just one element in common with each of the subsets $A, B, C, \cdots$.† To this postulate Peano objects that: one may not apply an infinite number of times an arbitrary law by virtue of which one correlates to a class some member of that class.‡ Here are two postulates by two different authorities; the postulates are contradictory, and each thinker is at liberty to adopt whichever appears to him the more convenient.

Lebesgue has recently pointed out the high degree of unattainability which characterizes the correspondence postulated by Zermelo.§ I should like to call attention to another awkwardness of the theory. Consider at first a finite set of elements, say $M(1, 2, 3, 4)$, and follow out the statement of Zermelo's

* Zermelo, *loc. cit.*, *Annalen*, 59, p. 516.

† Zermelo, *loc. cit.*, *Annalen*, 65, p. 110.

‡ Peano, *loc. cit.*, *Revista*, 8, p. 145. Peano had taken this position in an earlier communication in 1890, *Mathematische Annalen*, vol. 37, p. 210.

§ Lebesgue, "Contribution a l'étude des correspondances de M. Zermelo," *Bulletin de la Société Mathématique de France*, vol. 35, pp. 202-212.

fundamental theorem: Ist durch irgend ein Gesetz jeder nicht-verschwindenden Untermenge einer Menge M eines ihrer Elemente als "ausgezeichnetes Element" zugeordnet, so besitzt die Menge $\mathfrak{U}(M)$ aller Untermengen von M eine und nur eine Untermenge $\mathfrak{M}$ von der Beschaffenheit, dass jeder beliebigen Untermenge P von M immer ein und nur ein Element P_0 von $\mathfrak{M}$ entspricht, welches P als Untermenge und ein Element von P als ausgezeichnetes Element enthält. Die Menge M wird durch $\mathfrak{M}$ wohlgeordnet.* To avoid all the difficulties of following the author's deduction of this theorem, let it be assumed that the theorem is properly deduced.

In the first place it is necessary to construct the set $\mathfrak{U}(M)$ of all subsets of M. This is the set

1, 2, 3, 4, 12, 13, 14, 23, 24, 34, 123, 124, 134, 234, 1234.

The number of elements in this set is $4 + 6 + 4 + 1 = 15$. More generally if M had contained n elements, the set $\mathfrak{U}(M)$ would have contained

$$n + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} + \cdots + n + 1 = 2^n - 1$$

elements. Next it is to be supposed that in each of these sets some element has been picked out as ausgezeichnet. This may be accomplished by printing that element of each set in red. The number of ways in which it may be accomplished is 20736. In case the number of elements had been n, the number of ways would have been

$$(a) \qquad 2^{\frac{n(n-1)}{2!}} \cdot 3^{\frac{n(n-1)(n-2)}{3!}} \cdots\cdots (n-1)^n \cdot n.$$

The next step is the assertion of the existence of the set $\mathfrak{M}$ of the specified properties. This set will depend in large measure upon the particular way in which the ausgezeichnetes element of each set has been chosen. If it were agreed to print the smallest integer in each set in red, the set $\mathfrak{M}$ would be

1234, 234, 34, 4,

and the corresponding way in which M would be well ordered would be 1, 2, 3, 4. However, by the method outlined by Zermelo, loc. cit., page 110, it is clear that with a proper dis-

* Zermelo, *loc. cit.*, *Annalen*, 65, p. 108.

tribution of the red ink it would be possible to obtain a set $\mathbf{M}$ which would give M any one of its possible 4! arrangements. In like manner if M contained n elements, a proper selection of the red elements would give rise to an $\mathbf{M}$ which would ensure to M any of its $n!$ possible orders.

As a matter of fact, it may be seen from Zermelo's reasoning that $\mathbf{M}$ must contain M, and at least one of the subsets of M in which only one element of M is missing, also at least one of the subsets of this subset where just one additional element is missing, and so on. Thus suppose that $\mathbf{M}$ were picked in accordance with this principle as

1234, 134, 13, 3.

The corresponding ordering of M would be 2, 4, 1, 3, and it would not be hard to assign a red element in each subset of M on this basis. This assignment could probably be made in an enormous variety of ways; for the number of ways in which red elements may be selected is (a), in this case 20736, whereas the possible arrangements for M number only $n!$, in this case 24.

Thus in the case of the well-ordering of a finite set a very complicated mechanism has been introduced which suffices not only to order the set, but to order it in all possible ways and moreover to accomplish this ordering in an even vastly greater number of ways. One is tempted to say that not only has the author proved his proposition (for finite sets), but so much more than proved it as to render the conception of well-ordering almost worthless. In view of the fact that there are many who do not believe in the transfinite cardinals at all and that those who do believe find it no easy matter to escape error in manipulating these numbers, one must be very cautious in extending to infinite sets a method of reasoning analogous to that here employed with reference to finite sets. Some general considerations may, however, be worth notice in a tentative way.

Suppose that M is the ensemble of elements in the continuum. Let their number be $\aleph_1$.* The set $\mathbf{U}(M)$ according to those who believe in such things would have a greater number, say $\aleph_2$. The very introduction of this set into the reasoning on the well-ordering of the continuum implies the use of something

*I do not wish to imply the existence of the transfinite cardinals: the three aleph symbols here introduced may be regarded as notations for hypothetically existing entities.

which is more transfinite than is necessarily called for by the theorem to be proved. And this greater transfinitude must be in working order if Zermelo's correspondence is to be obtainable. What might be the order of infinitude which shall suffice for specifying the number (a) of ways in which the correspondence may be set up is a question which will be left to the reader. There are, however, indications which point with considerable assurance to the inference that if Zermelo's reasoning were followed to its (logical?) conclusion, it would result in establishing the fact that the continuum may be well ordered in something like $\aleph_1!$ ways. If so, is the theorem worth having? Does it mean anything?

To get a glimpse of the bearing of these questions it may be well to look at the problem of well-ordering the positive rational numbers. The method of ordering them by telling them off against the positive integers is well known and is valuable. Suppose Zermelo's method were followed. Here $\mathbf{U}(M)$ would be that set which is introduced in defining the continuum — a set of greater infinitude than M. Suppose next that the correspondence required by the author were set up, the set $\mathbf{M}$ picked out, and the ordering accomplished. In view of the evidence that this could all be accomplished (if at all) in a high infinitude of ways, is it evident that any more (and perhaps even less) has been accomplished than would have been accomplished by the simple statement that to order and well-order the positive rational numbers it is merely necessary to pick out one of them and set it down and then pick out another and set that down next to the former, and so on?*

To sum up this discussion I should say:

1° that Zermelo is right in his contention that he may add his postulate to the existing logical system; that to deny him that privilege would be to put an embargo on the development of logic and to assume a completeness of our logical system which is quite unwarranted in view of past developments and future possibilities:

2° that the use he makes of his principle is to render a hard thing harder, a transfinite condition more transfinite; that there still remains so much of an arbitrary nature in the determination of the correspondence as to make his theorem on well-ordering

* This idea has been stated by Borel, "Quelques remarques sur les principes de la théorie des ensembles," *Mathematische Annalen*, vol. 60, pp. 194–195. For comment, see the next footnote.

of little more value than the statement that the elements of any set may be plucked out seriatim and set in a row; that he applies this latter principle (except for the seriatim plucking) to a case which really transcends the original set in infinitude:

3° That the well-ordering of any set is of practically no significance and is quite worthless apart from an algorithm which accomplishes the ordering — an algorithm which shall not require an operation which transcends the cardinal number of the given set.*

It should be added that, entirely apart from the problem of well-ordering, the application of Zermelo's principle to the construction of a general theory of sets (Mengenlehre) along the lines he is now following is an interesting study which may contribute in no small measure toward the solution of some of the fundamental questions of logic and may easily result in the general adoption of some form of the principle which he has introduced explicitly and which has tacitly been used to a very considerable extent with no overwhelmingly fallacious results. It seems far from certain that we shall never need a more general principle of reasoning than that contained in the present acceptation of the scope of mathematical induction.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
BOSTON, MASS., *April*, 1908.

* In forming any conclusions concerning Zermelo's problem one must not overlook the keen remarks made on both sides of the question by several authors and published under the title "Cinq lettres sur la théorie des ensembles" in the *Bulletin de la Socié é Mathématique de France*, vol. 33, pp. 261–273. My conclusions here stated agree very largely with those of Borel and Baire in those lettres; and hence there would be little need of my making the restatement that I do, were it not for the fact that I feel that I have treated the matter somewhat more in detail and in a way less open to objection on the ground of subjectivity or prejudice. Hadamard pointed out in his first letter (and this has an application to the last statement in 2°) that there was a difference between picking out one element after another from an infinite set and setting them in order on the one hand and selecting an element from each one of an infinity of independent sets on the other hand. If that be granted (it is not granted in the present case by all who have commented on the subject), I still fail to see how there is any great difference between plucking one element after another from the continuum and selecting an element from each one of the subsets of the continuum, especially in view of the fact that the set of subsets is of greater number (Mächtigkeit) than the continuum. It is the application of the principle of selection to this set of greater number which particularly impresses me; and in addition to this, the so great freedom of choice which is to be found in the selection. The freedom appears quite sufficient to enable the well-ordering to be likened to plucking one element after another. So far as I am aware, these two fundamental considerations have not been taken up by previous writers. This, with the discussion of logic which occupies the first half of the paper, may perhaps be a sufficient justification for my present communication.

Postscript. — Since the above was sent to the printer, Schoenflies's report on Die Entwickelung der Lehre von den Punktmannigfaltigkeiten, 2ter Teil, has come to hand and should be included among the references cited in this paper. On page 36, paragraph 3, the author states a conclusion which is practically that of Borel in the *Annalen*, loc. cit., volume 60, namely, that Zermelo's proof can be characterized as establishing the equivalence of the principle of selection and the principle of well-ordering. That is to say, if a given set is well-ordered a correspondence such as Zermelo postulates may be set up between the subsets and the elements of the set by simply associating to each subset its first element, and Zermelo's proof shows that conversely if the principle of selection is admitted the given set may be well-ordered.

I am by no means sure that these statements bring the two principles into equivalence, as Borel and Schoenflies assume. It should be remembered that if a set is well-ordered we can actually associate to *any* subset an ausgezeichnetes element, whereas the postulate of selection requires us by an arbitrary process to associate to *all* the subsets an element. As far as finite sets go, the distinction between any and all is not so great as not to be readily overlooked, although logicians regard a postulate concerning any element of a set as far simpler than one concerning all elements. When, however, it is a question of infinite sets and particularly of non-enumerable sets the difference between any and all is vast, and many statements or proofs, unexceptionable when phrased in terms of any, are by no means acceptable when all replaces any.

In view of these considerations and the oft-repeated fact that Zermelo's principle must be applied to a set of greater number than the one to be ordered, I should seriously doubt the equivalence between his principle and that of ordering and should feel quite justified in claiming that it was logically and mathematically preferable to take the principle of ordering as a postulate instead of the principle of selection in the general form in which Zermelo states it.

E. B. W.

May, 1908.

SHORTER NOTICES.

Sur le Développement de l'Analyse et ses Rapports avec diverses Sciences: Conférences faites en Amérique. Par EMILE PICARD. Paris, Gauthier-Villars, 1905. 167 pp.

La Science moderne et son Etat actuel. Par EMILE PICARD. Paris, E. Flammarion, 1908. 301 pp.

PICARD'S lectures delivered here in America, three at the decennial of Clark University in 1899 and one at Saint Louis in 1904, were published soon after their delivery. Those in celebration of the decennial of Clark appeared in the volume devoted to that celebration and again in the *Revue générale des sciences* early in 1900. The lecture at Saint Louis had its place in the Proceedings of the congress of arts and sciences. It was not, however, until 1905 that the author collected the four lectures in a separate monograph readily accessible to all.

In the general introduction to the first lecture the author states: "Je lis toujours pour ma part . . . le *Bulletin* de la Société mathématique américaine . . . qui tient ses lecteurs au courant des travaux les plus récents." It is much to be feared that the present review is too tardy to come under this generous compliment. There is, however, in all that Picard writes, and particularly in these lectures, that finish and that wisdom which makes his work immediately a classic and compels, not at the time merely but for many years thereafter, the attention and interest of the student who would really be "au courant des travaux les plus récents."

The chief thoughts in the first lecture are grouped about the genesis and growth of the general conception of function. The subject takes one far and wide in the history of the development of mathematics and there is scarcely a reputable mathematician since Euler whose name does not deserve mention in the sketch; for although the functions of real and of ordinary complex variables naturally take most of the attention, the possibility of functions of other hypercomplex numbers and the various geometric fields therewith connected are not overlooked. That the author is not so exclusively concerned with the scientific side of his subject as to forget the fundamental pedagogic ideas, is brought into evidence by the two remarks: "Si Newton et Leibniz avaient pensé que les fonctions continues n'ont pas nécessairement une dérivée, ce qui est le cas général, le calcul

différentiel n'aurait pas pris naissance; . . . " and in reference to the extremes to which rigor may be pushed "ces spéculations raffinées ont même pénétré dans l'enseignement élémentaire, ce qui est, a mon avis, très regrettable." And such matters are not merely mentioned; they are treated as worthy of some elaboration.

The second lecture is on the theory of differential equations, whether partial or ordinary. Many interesting results are cited from the work of the previous decade, and not a few of them are due to the author or to others who have followed where he has led. One fact that is rarely seen in print is the discovery of Borel that every analytic integral of a partial differential equation with analytic coefficients may be expressed by a formula containing only one arbitrary function of a single real variable. In regard to ordinary equations, the work of Painlevé on equations which are not linear and of Poincaré on the configuration of the real solutions are among the foremost of the matters noticed. The author very naturally records a protest against the tendency to regard the theory of ordinary equations as a chapter in the theory of analytic functions.

That Picard would hardly subscribe to the motto "Es giebt nur Potenzreihen" appears evident from the fact that much of the discussion in the first two lectures is connected with the name of Fourier and that it is only in the third lecture that he comes to speak especially of analytic functions and of the various special functions which have been introduced into mathematics. After a preliminary reference to the origin of the theory of analytic functions and an extended mention of the recent work of the French school, among whom Mittag-Leffler must be included, on the various developments of these functions and on the special properties of integral functions, the author draws the general outlines of the automorphic functions, the new transcendents of Painlevé, the hyperautomorphic functions, and finally his own algebraic functions of two variables. Desiderata are not forgotten amid the mass of results already acquired.

The concluding lecture, that at Saint Louis, wherein the general development of mathematics in connection with its applications is reviewed from early times through the century of d'Alembert's bon mot "allez en avant, et la foi vous viendra" down to the present rigorous days, cannot be sub-reviewed: it must be read. One idea, however, which will appear again in

La Science moderne, may be noted: that of heredity as against non-heredity, of functional equations as against differential equations.

Modern science as it is is a subject so vast that no one person can hope to be an "Ausgelernter" in science alone, to say nothing of all other domains of thought, as might have been possible in the days of Faust. Yet if there is a place where such an accomplishment is most favored it is Paris, where the five academies flourish and where intellectual pursuits are centralized as they are nowhere else. Picard's book, La Science moderne, may well take its place among the fair flowers of this hothouse of thought.

In the introduction, after shutting out metaphysical considerations and limiting himself to a discussion of the methods of investigation which are essentially scientific, the author points out the constantly growing field of science and the constantly increasing correlation of different parts of the field, until "Il semble que dorénavant, dans la vie scientifique comme dans la vie sociale, l'association s'imposera de plus en plus. Tel travail ne pourra être effectué que par la collaboration d'un mathématicien et d'un physicien, et tel autre demandera le concours d'un chimiste et d'un physiologiste." This is a generalization and at the same time a reminiscence of the idea with which he closed his lectures at Clark — an idea which in practice is constantly and emphatically outraged by the training that is provided by some of our foremost schools as preparation for the doctorate — "En terminant, je me permettrai de donner un conseil aux étudiants mathématiciens qui m'ont fait l'honneur de m'écouter: je leur recommanderai de ne pas se cantonner trop tot dans des recherches spéciales. Il leur faut acquérir d'abord des vues générales sur les diverses parties de notre science, sans lesquelles leurs recherches risqueraient de rester stériles, et qui leur coûteraient plus tard un bien plus grand effort."

The first chapter is a summary of the lecture at St. Louis, explained so far as may be without technicalities and adapted to the comprehension of a more popular audience. The second chapter bears the title, the mathematical sciences and astronomy. There is much more in it than the title would indicate; for a start is made from the modern theories of number and of analysis and from some rapid indications concerning recent advances

in geometry, with especial reference to the theory of groups (it should be remembered that the new program of public instruction in France introduces the notion of groups into geometry at a very early stage) and of the popularly attractive non-euclidean geometries. And some further words on the development of analysis are inserted before the author finally settles down to celestial mechanics and astronomical physics. So much for the venerable sciences of mathematics and astronomy.

The next chapter logically comes to the subjects of mechanics and energetics. Here there is an account of mechanics from its early formative stages through the time of Newton; some mention of the deductive side of mechanics from that day to the present, with especial emphasis on the ideas of Hertz in regard to hidden motions and hidden masses; and then the statement and discussion of the problem of the mechanical explanation of the universe. There is a concluding section on energy, in which the laws of thermodynamics and the relation of mechanics to the wider fields of physics receive attention. The fourth chapter is on the physics of the ether. The way in which the author manages to set forth the interplay of optics with electricity and magnetism, to tell of the recent discoveries in cathode and X-rays, and to expound the chief phenomena and theories of the radioactive substances is to be admired and beheld with wonder.

That Picard is proceeding from the more abstract and definite toward the more phenomenological and less thoroughly correlated sciences may be seen from the titles of the remaining five chapters: material physics and chemistry, mineralogy and geology, physiology and biological chemistry, botany and zoology, and medicine and the microbic theories. Whether or not the first four chapters would be readily understood by persons who, though educated, were not specialists in mathematics or physics might at first be a serious question: but if the explanation there is as clear and interesting as it surely is in these five last chapters, which are certainly not directly in our province, the answer could only be in the affirmative. To read, in the author's delightful French, of the accomplishments in all these fields by such a variety of authorities as Moissan, Suess, Koch, Lehmann, Haeckel, Pasteur, Loeb, Marsh, de Vries, and many others whose names are less familiar, is highly interesting and leaves the same literary and romantic impression as Maeterlinck's L'Intelligence des fleurs, less the fantasy — but would

scarcely find a proper place for review in this periodical. Besides, it should appear that even if one man did write of all these matters, one reviewer could not be expected to treat all of them in other than the perfunctory way in which almost all offerings of scientific work are "reviewed" in the daily or weekly press.

E. B. WILSON.

Algebraic Equations. By G. B. MATHEWS, M.A., F.R.S., Fellow of St. John's College. Cambridge University Press, 1907. 64 pp.

THIS is one of a half dozen "Cambridge Tracts in Mathematics and Mathematical Physics," issued by the Cambridge University Press. In this tract the author is confronted with the problem of giving an exposition of the Galois theory of equations within the compass of about sixty pages. This limitation makes it necessary for him to present the subject more or less in outline and to confine himself to a very few illustrative examples. An outline presentation, prepared by an eminent author, is certain to bring out in bold relief interesting view points. Such is the case in this booklet. And yet we are of the opinion that the real value of this book to beginners would have been enhanced by more abundant illustration and a somewhat fuller detail of explanation.

To save space, the author does not put down a definition in a sentence by itself; the definition is to be inferred from a condensed statement made as part of a sentence occurring perhaps in the body of a demonstration. Thus, the definition of an intransitive group (page 14) is given in course of a proof, as follows: "First suppose G is intransitive: this means that a certain number of roots $x_1, x_2, \cdots, x_r$ $(r < n)$ are only interchanged among themselves by the substitutions of G." Less easily comprehended are the definitions of simple groups and self-conjugate factor groups, similarly interpolated on pages 16 and 17. Despite the effort to secure extreme condensation, there occur redundancies, such as "absolutely undetermined" (page 2), "absolutely unaltered" (page 57), "perfectly definite" (page 6). These are less objectionable in oral exposition than in a printed outline.

Here and there are evidences of hasty composition. Thus, the tract is encumbered with some heterogeneous terminology. The author speaks in different places of the "arithmetically

rational," and the "rational in the ordinary sense," also of a "definite numerical irrationality" and an "irrational quantity." The terms "rational" and "rational function" are not always accompanied with an explicit statement of the field of rationality, so that extra care must be exercised by the reader, lest he misinterpret the author. On page 5 is proved the theorem that any rational function of the roots of a given equation $f(x) = 0$ can be expressed as a rational function of any one of the roots of its complete galoisian resolvent. This is done at a time when the meaning of the term "rational" has not, as yet, been extended to apply to different fields. However, the generalization is made on the next page. As a corollary to this theorem, it is shown that all the roots of a complete galoisian resolvent may be expressed as rational functions of any one of them; an equation having this property is called a normal equation. As this complete galoisian resolvent may be reducible in the field defined by the coefficients of the given equation, it would follow that a normal equation may be reducible — a definition that is both undesirable and contrary to the usage of writers on the Galois theory.

An unfortunate phraseology occurs on pages 2, 3 and 4, in connection with "arbitrary" and "numerical" coefficients. It leaves the impression that equations of higher degrees than the fourth are algebraically unsolvable only when the coefficients are "arbitrary." "If $n < 5$ and the coefficients are left arbitrary, it is possible to construct an explicit algebraic function of the coefficients which is a root of the equation. For $n > 4$, this is no longer the case." As if this statement did not apply at all to equations with numerical coefficients, the author proceeds to say: "When the coefficients are numerically given," the roots can be found by trial or approximation, although the main problem with him is this: "Given a particular equation with numerical coefficients, it is required to find the simplest set of irrational quantities such that all the roots of the given equation can be expressed as finite rational functions, in an explicit form, of the set of irrationals." Returning to arbitrary coefficients, he gives then a preliminary discussion of the cubic, quartic and quintic. Nowhere on these pages does he convey the idea that numerical equations may be algebraically unsolvable, nor does he here make plain that, besides the extreme cases of coefficients that are all "arbitrary" or all "numerical," there are intermediate assumptions which demand our

attention. Needless to say, no one knows the facts in question better than our distinguished author, and it is a pity that he did not take time to express himself more precisely. Later the author drops the term "arbitrary coefficients" and speaks of "coefficients represented symbolically" (page 11).

The foundation of the Galois theory covers in this tract the first twenty-nine pages; then eleven pages are given to cyclic equations, four pages to abelian equations and ten pages to metacyclic equations. The last chapter is upon "Solution by standard forms" and contains, on six pages, in outline, the treatment of the quintic by means of the icosahedral irrationality. This outline can be used with interest in conjunction with Klein's Ikosaeder.

FLORIAN CAJORI.

Introduzione alla Geometria Proiettiva degli Iperspazi. By E. BERTINI. Pisa, Spoerri, 1907. 425 pp.

THE subject of geometry of n dimensions has undergone a noteworthy advancement during the last few decades. In this advancement the Italian geometers have taken a very prominent part. It is therefore fitting that this able work on projective geometry of n dimensions should be the work of an Italian.

The author avoids the difficulties of spacial intuition by starting from a purely arithmetical definition of a point and of a space. There exist, in ordinary geometry, numerous examples of systems of entities which can be put into one to one algebraic correspondence, without exceptional elements, with the values of the ratios of a system of parameters. The properties of such systems are of two kinds, those depending on the particular nature of the entities and those depending only upon the fact of the correspondence with the group of values of the parameters. The latter properties the author considers to form the subject matter of projective geometry of n dimensions. A point is therefore defined as a set of ratios of $n + 1$ parameters $x_0, x_1, \cdots, x_n$, and a space of n dimensions as "the totality formed by all the sets of values (real and complex) of these ratios."

The first two chapters are devoted to a discussion on this basis of the ideas of subordinate spaces, section, projection, etc. The possibility of a satisfactory synthetic treatment is also pointed out, however, and a number of references to such treatments are given.

The next three chapters, of about eighty-four pages, are de-

voted to the discussion of projective and dualistic correspondences. Chapters six and seven, containing each about twenty pages, treat of the properties of quadrics and systems of quadrics. Chapter eight, of twenty-five pages, is devoted to the discussion of hypersurfaces and chapter nine, of thirty pages, to the discussion of algebraic spreads in general. In chapters ten and eleven, containing about fifty pages, an able and interesting discussion of linear systems of hypersurfaces is given. Chapter twelve, of fifteen pages, treat of rational curves, with especial attention to rational normal curves. Chapter thirteen, containing twenty-three pages, treats of rational ruled surfaces. In chapter fourteen, of about twenty pages, the author discusses the relations of linear systems of plane curves to two dimensional spreads in space of n dimensions. Chapter fifteen, of twenty-five pages, is devoted to a thorough discussion of the surface of Veronese and its three dimensional projection, the Steiner surface.

Of especial importance is the appendix of about fifty pages. This is divided into three chapters. The first deals with branches of algebraic curves, the second with quadratic transformations and the third with correspondences on one dimensional spreads.

It will be seen that the book covers only a portion of the field of geometry of n dimensions; but the subjects treated are discussed with a great deal of thoroughness. The treatment is in fact, in some cases, rather profuse. It would, perhaps, have been better to omit some of the minor discussions except for a reference to where the treatment may be found.

A regretable feature of the book, from the standpoint of the general reader, is the scarcity of references. This lack is recognized by the author himself, but he points out that the book is intended primarily for undergraduates in the Italian universities, for whom an extensive bibliography did not seem so necessary.

The reader will find in no other place so complete and comprehensive an exposition of our present knowledge of projective geometry of n dimensions as is given in this book. It is clearly and logically written and unusually free from errors. It will doubtless stimulate a much wider interest in this field of research.

C. H. SISAM.

Analytische Geometrie des Punktes, der geraden Linie und der Ebene. Von OTTO STAUDE. Leipzig, B. G. Teubner, 1905. viii + 447 pp.

THIS volume is offered as an introduction to the theory of surfaces of the second order, which the publishers requested Professor Staude to write in connection with the Encyklopädie der mathematischen Wissenschaften. It is largely concerned with the various systems of coordinates used in the study of points, lines, and planes. Each subject is based upon cartesian coordinates, from which the other systems are derived.

The text is divided into three parts: the first treats of ranges of points and plane pencils; the second deals with the point, line, triangle, and quadrangle; the third discusses configurations of lines and planes. The subject matter is not new; the method is elegant.

Two characteristic features of this excellent work are the thoroughness with which the fundamental notions have been discussed, and the large place given to the principle of duality.

The historical and critical notes of the appendix are exceedingly valuable. The author indicates the important contributions of various geometers, states the sources of new conceptions, indicates parallel lines of development, and gives a multitude of references.

G. N. BAUER.

Problems in Strength of Materials. By WILLIAM KENT SHEPARD. Ginn and Co. viii + 72 pp.

THE purpose of this book, another of the Yale series of mathematical texts, is to furnish a large variety of problems to supplement and illustrate the theory of the strength of materials. The author believes that, in order to obtain a working knowledge of any scientific subject, the average student should solve numerous problems illustrating the application of the theory, and that the present text-books on the strength of materials do not furnish a sufficient number.

The 568 problems proposed cover fully the field of the subject, as usually taught in scientific and technical schools. The fundamental subjects of tension, compression, shear, and elastic deformation are first considered; after which the problems are mainly those of the design and investigation of cylinders and spheres, riveted joints, the various types of beams and columns, plates, and shapes to resist torsion and combined stresses.

The use of the factor of safety in designing is emphasized throughout.

The purpose of the author evidently is to make the problems as practical as possible; this he might have realized more fully if, for instance in the design of columns, he had given specific illustrations of such shapes actually designed and erected. Again, while the figures throughout the book are excellent and make their point, they would have been more efficient if, for example, in the case of riveted joints, they had represented actual sections, correct from the standpoint of engineering practice.

The typography of the book is up to the high standard of all of the texts of the Yale series. Tables of constants and data usually included in engineers' handbooks are given at the back.

ERNEST W. PONZER.

Das 200-jährige Jubiläum der Dampfmaschine, 1706–1906. Eine historisch-technisch-wirtschaftliche Betrachtung. Von KURT HERING, Ingenieur. Leipzig, Berlin, B. G. Teubner, 1907. Pp. 58.

THIS pamphlet is not put forth as a history of the steam-engine. Such a history could not very well be written within so small a compass. Prepared at the 200th anniversary of Denis Papin's invention, as set forth in his "Ars nova" (1707), the pamphlet devotes itself mainly to an exposition of the ideas and experiments of this noted French physicist, who for many years resided in Germany. Papin is looked upon by the author as the theoretical inventor ("geistige Erfinder") of the steam engine. It is unfortunate that an exact date (1706) is set up in the title of the publication for the invention. This date fixes Papin as the inventor, but such a claim the author does not really make in the body of the book. We translate from his introduction: ". . . but who was the inventor of this most important machine of modern times? Not as the work of one only, but as the product of many scholars and practical men of the most different nations, does the steam engine come down to us." One historical point, discussed by Hering, needs emphasis here, namely that Papin never constructed a steamboat and that the boat on which he travelled on the river Fulda from Kassel to Münden on September 24, 1707, though embodying the novel idea of paddle wheels, was not driven by steam. Since 1690 Papin had been con-

sidering the idea of applying steam power to boats, but in the description of a boat under construction, given in a letter to Leibniz (March 13, 1704), Papin says: "Je n'ay point préparé celui ci pour y emploier la force du feu: parceque ce n'est pas à moi d'entreprendre trop des choses à la fois." In Robert H. Thurston's History of the growth of the steam engine (New York, 1893), we read (pages 224, 225) that Papin propelled his boat by his "steam engine" and that "an account of his experiments is to be found in manuscript in the correspondence between Leibniz and Papin, preserved in the Royal Library at Hanover." As Thurston does not give the date of the letter, nor quote from it, while Hering supports his contention also by quoting E. Gerland, who for thirty years has been engaged in editing the correspondence between Leibniz, Papin, and Huygens and the Leibnizian manuscripts on mechanics, and who stoutly combats the conclusion that Papin ever built himself a boat driven by steam, it would seem that Thurston must be in error. Were there any real support to the story of Papin's steamboat, then surely the Germans would be the last to deny it.

The pamphlet closes with a chapter embodying statistics and charts to show the effect of the introduction of the steam engine upon industrial life, particularly in Germany.

FLORIAN CAJORI.

NOTES.

AT the April meeting of the Council of the AMERICAN MATHEMATICAL SOCIETY, Professor E. B. VAN VLECK was re-elected a member of the Editorial Committee of the *Transactions*.

THE April number (volume 9, number 3) of the *Annals of Mathematics* contains the following papers: "A necessary condition that all the roots of an algebraic equation be real," by O. D. KELLOGG; "The equilibrium of a heavy homogeneous chain in a uniformly rotating plane," by E. B. WILSON; "The continuity of the roots of an algebraic equation," by J. L. COOLIDGE; "On the differentiation of definite integrals," by W. F. OSGOOD; "Note on the convergence of a sequence of functions of a certain type," by H. S. BUCHANAN and T. H. HILDEBRANDT; "On the inverse problem of the calculus of

variations," by G. A. BLISS; "A geometrical interpretation of the generalized law of the mean," by C. N. HASKINS; "On the torsion of a curve," by P. L. SAUREL.

AT the meeting of the London mathematical society held on April 30 the following papers were read: By E. W. HOBSON, "On a general convergence theorem and the theory of the representation of a function by series of normal functions"; by G. H. HARDY, "On the multiplication of series"; by F. H. JACKSON, "On q integration and differential equations."

THE Guccia medal was awarded Monday, April 6, at the first session of the fourth international congress of mathematicians. No one of the competing memoirs was deemed worthy of the prize, and according to the conditions (BULLETIN, volume 11, page 167) it was to be given to the author of the most important memoirs in algebraic geometry during the interval from November, 1904, to July, 1907. The medal was consequently unanimously awarded to FRANCESCO SEVERI, professor of mathematics at the University of Padua.

AT the Rome meeting of the fourth international congress of mathematicians a committee of organization was appointed to investigate the teaching of mathematics in various countries and report at the next congress in 1912. This committee, consisting of Professor F. KLEIN, of the University of Göttingen, Professor A. G. GREENHILL, of the Ordinance College, Woolwich and Professor H. FEHR, of the University of Geneva, was charged with the duty of selecting the members of the general committee after consulting with the councils of the leading mathematical societies of the various countries.

THE Deutsche Mathematiker Vereinigung will hold its annual meeting in affiliation with the eightieth convention of the society of German naturalists and physicians at Cologne during the week beginning September 20. While papers on mechanics will form the chief subject for discussion, memoirs on other subjects are welcome, and should be reported to the secretary, Professor A. KRAZER, of Karlsruhe, at once. Professor F. KLEIN, of Göttingen, is president during the present year.

THE fourth volume (1759–1799) of Cantor's Geschichte der Mathematik has recently appeared, the first copy having been laid before the international congress at Rome.

THE complete report of the investigation made by the *Enseignement Mathématique* and the department of psychology of the University of Geneva regarding the methods of work of various mathematicians has been republished in a pamphlet of 126 pages by the firm of Gauthier-Villars in Paris under the title Enquête de l'Enseignement Mathématique sur la méthode de travail des mathématiciens.

FROM a pamphlet recently issued by the departments of mathematics and mathematical astronomy of the University of Chicago it appears that from the opening of the University in 1892 to October, 1907, 1194 students pursued in these departments studies above and including the calculus. Of these 689 were registered in the graduate school. The number of major courses offered from the calculus upward was 624. The master's degree was conferred on 50 candidates, the doctor's degree on 33.

A NUMBER of mathematical books from the library of the late Professor Heinrich Maschke are offered for sale at advantageous prices. A list of the books can be obtained from Professor Oskar Bolza, University of Chicago, to whom correspondence should be addressed.

DR. A. TAUBER, of the University of Vienna, has been promoted to an assistant professorship of mathematics.

DR. R. ROTHE, of the technical school at Charlottenburg, has been appointed professor of mathematics at the academy of mines at Clausthal.

PROFESSOR J. WEINGARTEN, of the University of Freiburg, has been declared professor emeritus, with the rank of full professor.

PROFESSOR M. CANTOR, of the University of Heidelberg, has retired from active teaching with the title of honorary professor.

PROFESSOR K. HENSEL, of the University of Marburg, has been elected a member of the academy of sciences of Halle.

PROFESSORS JOSEPH LARMOR and W. E. STORY have been elected members of the National Academy of Sciences.

PROFESSOR MAX MASON, of Yale University, has been appointed associate professor of mathematics at the University of Wisconsin.

AT the University of Michigan, Mr. W. H. BUTTS has been appointed to a junior professorship of mathematics.

AT the Michigan Agricultural College Dr. CARL GUNDERSEN has been promoted to an assistant professorship of mathematics.

DR. A. G. HALL, of Miami University, has been appointed to a professorship of mathematics at the University of Michigan.

MR. H. F. HART has resigned his instructorship in Syracuse University to accept the position of teacher of mathematics in the Montclair, New Jersey, high school.

PROFESSOR PETER FIELD, of the University of Michigan, has been granted a leave of absence for next year, to study in Europe.

PROFESSOR G. N. BAUER, of the University of Minnesota, has received leave of absence until autumn of 1909, and will study at the University of Göttingen.

DR. G. PICCIATI, who had just been appointed professor of theoretical mechanics at the University of Bologna, died March 11 at the age of 39 years.

PROFESSOR S. M. KEPINSKI, of the technical school at Lemberg, died March 28 at the age of 40 years.

PROFESSOR W. SCHEIBNER, of the University of Leipzig, died April 8, at the age of 82 years.

PROFESSOR A. MAYER of the University of Leipzig, editor of the *Mathematische Annalen*, died April 11, at the age of 69 years.

RECENT catalogues of second-hand mathematical books: Henry Sotheran and Co., 140 Strand, London, catalogue No. 682, about 4,600 titles in mathematics and sciences. — C. Lang and Co., Via Bocca di Leone, 13, Rome, catalogue No. 3, about 800 titles in general science. — Buchhandlung Gustav Fock, Schlossgasse 7–9, Leipzig, catalogue No. 331, journals and collections, mathematics, physics, and astronomy, 22 pp.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BACHL (F.). Ueber dreifach orthogonale Flächensysteme. (Progr.) Wien, 1907. 8vo. 13 pp.

BONOLA (R.). Die nichteuklidische Geometrie. Historisch-kritische Darstellung ihrer Entwickelung. Autorisierte deutsche Ausgabe besorgt von H. Liebmann. Leipzig, Teubner, 1908. M. 5.00

BÖRNER (E.). Ueber irreducible Nichtquaternionen-Systeme mit sechs Haupteinheiten. (Progr.) Mährisch-Ostrau, 1907. 8vo. 44 pp.

BROMWICH (T. J. I'A.). An introduction to the theory of infinite series. London and New York, Macmillan, 1908. 8vo. 13 + 511 pp. Cloth. $4.50

BURKHARDT (H.). Funktionentheoretische Vorlesungen. Vol. I. 2tes Heft: Einführung in die Theorie der analytischen Funktionen einer komplexen Veränderlichen. 3te, durchgesehene und vermehrte Auflage. Leipzig, Veit, 1908. 8vo. 12 + 262 pp. Cloth. M. 8.00

CAIRNS (W.). Die Anwendung der Integralgleichungen auf die zweite Variation bei isoperimetrischen Problemen. (Diss.) Göttingen, 1907. 8vo. 68 pp.

CANTOR (M.). Vorlesungen über Geschichte der Mathematik. Vol. IV. Von 1759 bis 1799. 4te Lieferung. Abschnitt XXVI: G. Vivanti, Infinitesimalrechnung. Leipzig, Teubner, 1908. 8vo. Pp. 643-882. M. 7.00

DOEHLEMANN (K.). Geometrische Transformationen. 2ter Teil: Die quadratischen und höheren, birationalen Punkttransformationen (Sammlung Schubert, XXVIII). Leipzig, Göschen, 1908. 12mo. 8 + 328 pp. Cloth. M. 10.00

DURÈGE (H.). Theorie der elliptischen Funktionen. 5te Auflage, neu bearbeitet von L. Maurer. Leipzig, Teubner, 1908. 8vo. 8 + 436 pp. Cloth. M. 11.00

GARBIERI (G.). Geometria analitica: riassunto di lezioni date nell'università di Genova. Parte II (Luoghi di 2° grado in coordinate cartesiane con numerosi esercizi). Torino, Paravia, 1908. 8vo. 76 pp. L. 4.00

GÜNTHER (S.). Geschichte der Mathematik. 1er Teil: Von den ältesten Zeiten bis Cartesius (Sammlung Schubert, XVIII). Leipzig, Göschen, 1908. 12mo. 7 + 427 pp. Cloth. M. 9.60

HARTL (H.). Erste Einführung in die Elemente der Differential- und Integralrechnung und deren Anwendung zur Lösung praktischer Aufgaben. Wien, Deuticke, 1908. 8vo. 3 + 59 pp. M. 1.20

JANZEN (O.). Ueber einige stetige Kurven, über Bogenlänge, linearen Inhalt und Flächeninhalt. (Diss.) Königsberg, 1907. 8vo. 70 pp.

KHANDRIKOV (M.). Infinitesimal Analysis. Vol. III. (Russian.) St. Petersburg, 1907. 8vo. M. 12.00

KISCHKE (R.). Ueber Fehlerabschätzung bei unendlichen Produkten und deren Anwendungen. (Diss.) Königsberg, 1908. 8vo. 83 pp.

KÜBEL (K. G. J.). Anwendungen einer anschaulichen Darstellung des Imaginären. (Diss.) Giessen, 1907. 8vo. 27 pp.

LALESCO (T.). Sur l'équation de Volterra. (Thèse.) Paris, Gauthier-Villars, 1908. 4to. 392 pp.

MAURER (L.). See DURÈGE (H.).

SCHEIBNER (W.). Beiträge zur Theorie der linearen Transformationen, als Einleitung in die algebraische Invariantentheorie. Leipzig, Teubner, 1907. 8vo. 250 pp. M. 10.00

SCHUBERT (H.). Niedere Analysis. 1er Teil: Kombinatorik, Wahrscheinlichkeitsrechnung, Kettenbrüche und diophantische Gleichungen. 2te Auflage. (Sammlung Schubert, V.) Leipzig, Göschen, 1908. 12mo. 4 + 181 pp. Cloth. M. 3.60

VIVANTI (G.). See CANTOR (M.).

II. ELEMENTARY MATHEMATICS.

ARBES (J.). Ueber Erfindung, Gestaltung und Wertschätzung der Logarithmen. (I.) (Progr.) Smichow, 1907. 8vo. 22 pp.

BARINGER (W.). Was muss man von der Arithmetik und der Algebra wissen? 2te Auflage. Berlin, Steinitz, 1907. 8vo. 71 pp. M. 1.00

BARONI (E.). Algebra e trigonometria, ad uso dei licei, secondo gli ultimi programmi. Vol. I (Primo anno di liceo). Firenze, Bemporad, 1908. 16mo. 6 + 363 pp. L. 2.75

BORTOLOTTI (E.). Nozioni pratiche di geometria, per le scuole complementari. 3a edizione, accuratamente riveduta. Roma, 1908. 8vo. 174 pp. L. 1.75

BOURDON. Eléments d'algèbre. 20e édition, revue et annotée par E. Prouhet. Paris, Gauthier-Villars, 1907. 8vo. 12 + 657 pp. Fr. 8.00

BRUNO (G. M.). Curso superior con numerosos ejercicios y nociones de agrimensura, levantamiento de planos y nivelación. Paris, 1908. 16mo. 400 pp.

CHAILAN (E.). Géométrie à l'usage des élèves de l'enseignement secondaire. Premier cycle, divisions A et B; second cycle, divisions A et B (programme de 1905). 4e édition. Paris, Poussielgue, 1908. 16mo. 8 + 240 pp.

DREYFUS (L.). See FORT (L.).

FENKNER (H.). Lehrbuch der Geometrie für den Unterricht an höheren Lehranstalten. (In 3 Teilen.) 3er Teil: Ebene Trigonometrie. Nebst einer Aufgabensammlung. Berlin, Salle, 1908. 8vo. 4 + 102 pp. M. 1.60

FORT (L.) et DREYFUS (L.). Eléments de géométrie, conformes aux programmes du 27 juillet, 1905. Classes de cinquième B et de quatrième A, avec de nombreux exercices, figures et problèmes. 2e édition. Paris, Paulin, 1908. 18mo. 8 + 215 pp. Fr. 2.50

HÄRING (G.). Lehrbuch der Geometrie für die Oberstufe der höheren Lehranstalten und zum Selbstunterricht. München, Oldenbourg, 1908. 8vo. 5 + 96 pp. M. 1.30

LOEWENBERG (G.). Was muss man von der sphärischen Trigonometrie und der Astronomie wissen? Berlin, Steinitz, 1907. 8vo. 77 pp. M. 1.00

LUDWIG (F. K.). Das logarithmische Rechnen. Leichtfassliche Darlegungen über das Wesen, die Berechnung und Anwendung der Logarithmen nebst zahlreichen Beispielen mit ausführlichen Lösungen. Zum Selbstunterricht bearbeitet. 2te Auflage. Komotau, Benker, 1908. 8vo. 53 pp. M. 1.30

MEHLER (F. G.). Hauptsätze der Elementar-Mathematik zum Gebrauche an höheren Lehranstalten. Neubearbeitet von A. Schulte-Tigges. Ausgabe A. 25ste Auflage des Stammbuches. Berlin, Reimer, 1908. 8vo. 12 + 280 pp. M. 2.40

PROUHET (E.). See BOURDON.

SCHULTE-TIGGES (A.). See MEHLER (F. G.).

III. APPLIED MATHEMATICS.

BOLTZMANN (L.). Vorlesungen über Maxwell's Theorie der Elektricität und des Lichtes. 2 Teile. 2ter unveränderter Abdruck. Leipzig, Barth, 1908. 8vo. M. 6.00

CAMPBELL (N. R.). Modern electrical theory. Cambridge, University Press, 1908. 8vo. 232 pp. Cloth. $2.25

D'OCAGNE (M.). See LECORNU (L.).

ENACHE (M.). Hydrodynamique. Contribution à la théorie de l'écoulement sur les déversoirs en mince paroi et à nappe voyée en dessous. (Thèse.) Paris, Gauthier-Villars, 1908. 4to. 37 pp.

JEANS (J. H.). The mathematical theory of electricity and magnetism. Cambridge, University Press, 1908. 8vo. 8 + 536 pp. Cloth. $4.50

JORDAN (H.). Zur Kenntnis der elastischen Nachwirkung bei kleinen gleichförmigen Biegungen homogener Metallstäbe. (Diss.) Göttingen, 1907. 8vo. 86 pp.

KOZÁK (J.). Grundprobleme der Ausgleichungsrechnung nach der Methode der kleinsten Quadrate. Vol. II. 1er Teil: Theorie des Schiesswesens auf Grundlage der Wahrscheinlichkeitsrechnung und Fehlertheorie. Wien, Fromme, 1908. 8vo. 21 + 400 pp. M. 16.00

LEBON (E.). Géométrie descriptive et cotée. (Classe de mathématiques A et B.) Paris, Delalain, 1908. 180 pp. Fr. 4.00

LECORNU (L.). Dynamique appliquée (Encyclopédie scientifique, Bibliothèque de mécanique appliquée et Génie, dirigée par M. d'Ocagne). Paris, Doin, 1908. 550 pp. Fr. 5.00

MATRICULATION mechanics papers. London, Clive, 1908. 8vo. 74 pp. 1s.6d.

WOTRUBA (R.). Technische Mechanik der starren, flüssigen und ausdehnsam-flüssigen Körper. Kurzgefasstes Hand- und Hilfsbuch für Techniker und Studierende an mittleren technischen Lehranstalten sowie zum Selbstunterricht. Leipzig, Voigt, 1908. 8vo. 10 + 134 pp. M. 5.50